Witten's Black Hole model: Symmetries and Integrability

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Outline

1. Introduction

2. Lagrangian Description

3. Chiral structure and integrability

4. Reduction

Bibliography

1.1 Introduction

- 2d CFT
- Classically integrable
- Well-defined scattering
- similarity to other integrable CFT

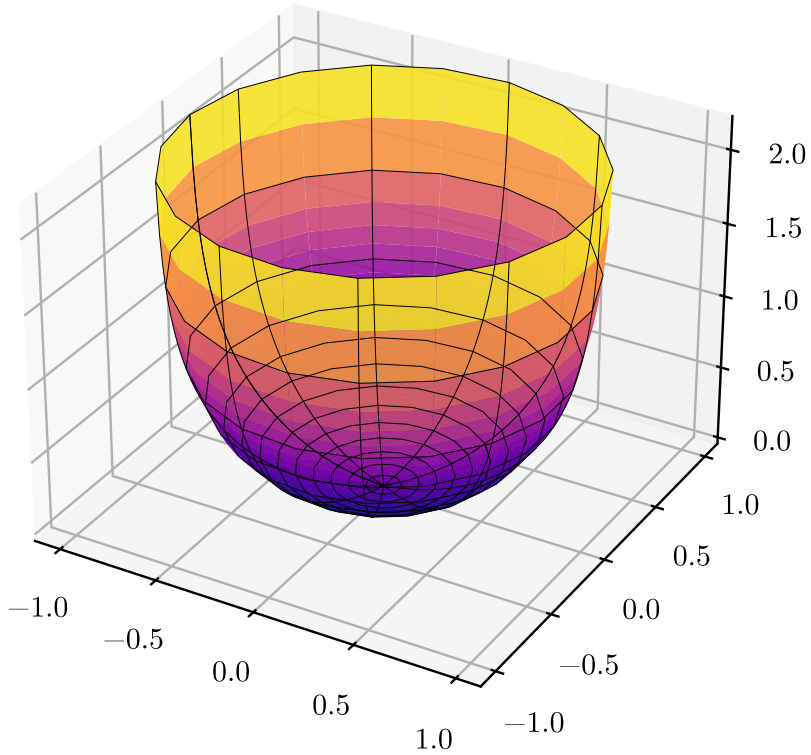


Figure 1: “Witten’s cigar”

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2.1 The Lagrangian

We start with the Lagrangian

$$\mathcal{L} = \frac{k}{2}\frac{\partial u \bar{\partial} u^* + \partial u^* \bar{\partial} u}{1 + uu^*}$$

- u is complex valued $u(x, \bar{x}) = u_1 + iu_2$, $u^*(x, \bar{x}) = u_1 - iu_2$
- k is a positive real number
- $\partial = \frac{\partial}{\partial x}, \bar{\partial} = \frac{\partial}{\partial \bar{x}}$.

This can be interpreted as a Sigma model with metric

$$ds = \frac{k}{2}\frac{du du^*}{1 + uu^*}$$

2.2 Equations of motion

The equations of motion are:

$$\partial\bar{\partial}u = u^* \frac{\partial u \bar{\partial} u}{1 + uu^*}$$

and its compplex conjugate. similar to E.O.M. of the $O(3)$ sigma model

$$\partial\bar{\partial}u = 2u^* \frac{\partial u \bar{\partial} u}{1 + uu^*}$$

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3.1 Stress-Energy tensor

2d CFT $\Longrightarrow$ off-diagonal Stress-energy tensor

$$T_+^+ = T_-^- = 0$$

$$T_+^- = \frac{\partial u \partial u^*}{1+uu^*} \qquad T_-^+ = \frac{\bar{\partial} u \bar{\partial} u^*}{1+uu^*}$$

with chirality conditions

$$\bar{\partial} T_+^- = 0 \qquad \partial T_-^+ = 0$$

$$\Downarrow$$

$$T_+^- \equiv T(x) \qquad T_-^+ \equiv \bar{T}(\bar{x})$$

3.2 Additional Chiral structure

$$S = \frac{\partial^2 u}{\partial u} - \frac{u\partial u^*}{1+uu^*} \qquad \bar{S} = \frac{\bar{\partial}^2 u}{\bar{\partial} u} - \frac{u\bar{\partial} u^*}{1+uu^*}$$

From the equations of motion we have

$$\bar{\partial} S = 0 \qquad \partial \bar{S} = 0$$

provides integrability of the model by rewriting E.O.M as

$$\partial^2 u = S\partial u + Tu \qquad \bar{\partial}^2 u = \bar{S}\bar{\partial} u + \bar{T} u$$

3.3 Integrability in general

- Hamiltonian Reduction
- Chirality conditions
- Zero Curvature Representation(Lax pair)

General solution

$$u = \Psi_1(x)\bar{\Psi}_1(\bar{x}) + \Psi_2(x)\bar{\Psi}_2(\bar{x}) = e^{2i\mu_0\tau}\langle \boldsymbol{t} f_L(x) f_R(\bar{x})\rangle$$

Monodromies

- Hyperbolic $\longrightarrow$ Scatterings
- Elliptic $\longrightarrow$ Bound states
- Parabolic

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4.1 the $O(3)$ sigma model

The $O(3)$ sigma model Lagrangian

$$\mathcal{L} = 2\partial \boldsymbol{q} \bar{\partial} \boldsymbol{q} + \frac{1}{2}\lambda(q^2 - 1)$$

$$\Downarrow$$

$$\partial\bar{\partial}\boldsymbol{q} - \left(\partial \boldsymbol{q} \cdot \bar{\partial}\boldsymbol{q}\right)\boldsymbol{q} = 0$$

stereographic projection with $u = u_1 + iu_2$

$$q^1 = \frac{2u_1}{1+uu^*} \qquad q^2 = \frac{2u_2}{1+uu^*} \qquad q^3 = 1 - \frac{2}{1+uu^*}$$

4.2 Stereographic projection

In stereographic projection coordinates $(q^1, q^2, q^3) \to (u^1, u^2)$

$$\mathcal{L} = 4\frac{\partial u \bar{\partial} u^* + \partial u^* \bar{\partial} u}{(1+uu^*)^2}$$

$$\Rightarrow \partial\bar{\partial} u = 2u^* \frac{\partial u \bar{\partial} u}{1+uu^*}$$

$$\partial\bar{\partial} u^* = 2u \frac{\partial u^* \bar{\partial} u^*}{1+uu^*}$$

4.3 Pohlmeyer reduction

The $O(3)$ model $\longrightarrow$ sine-Gordon equation

$$\partial\bar{\partial}\varphi = \sin(\varphi)$$

this is done by **Pohlmeyer reduction**.

From chirality conditions

$$\bar{\partial}T_+^- = 0 \quad \partial T_-^+ = 0 \Rightarrow T_+^- \equiv T(x) \quad T_-^+ \equiv \bar{T}(\bar{x})$$

set $T = \bar{T} = 1$ using a conformal transformation, leaving

$$\partial \boldsymbol{q} \cdot \bar{\partial}\boldsymbol{q} = 4\frac{\partial u\bar{\partial}u^* + \partial u^*\bar{\partial}u}{(1+uu^*)^2} = \cos(\varphi)$$

Can we do something similar to $SL(2,\mathbb{R})/U(1)$?

4.4 The problem with the original approach

In the $O(3)$ reduction we get

$$\partial \boldsymbol{q} \cdot \bar{\partial} \boldsymbol{q} = 4 \frac{\partial u \bar{\partial} u^* + \partial u^* \bar{\partial} u}{\left(1 + u u^*\right)^2} = \cos(\varphi)$$

which is helpful in the starting coordinates, because

$$\partial \bar{\partial} \boldsymbol{q} - \left(\partial \boldsymbol{q} \cdot \bar{\partial} \boldsymbol{q}\right) \boldsymbol{q} = 0 \Rightarrow \partial \bar{\partial} \boldsymbol{q} - \cos(\varphi) \boldsymbol{q} = 0$$

but for the stereographic projection coordinates

$$\mathcal{L} = 4 \frac{\partial u \bar{\partial} u^* + \partial u^* \bar{\partial} u}{\left(1 + u u^*\right)^2} = \cos(\varphi)$$

which **does not show up explicitly in the E.O.M.**

4.5 A more modern approach

- A group theoretic approach
- relies on reducing the Lax equations

$$\partial\Psi = U(\lambda)\Psi, \quad \bar{\partial}\Psi = V(\lambda)\Psi$$

  for an auxiliary field Ψ where $U, V \in \mathfrak{sl}(2, \mathbb{R})$
- the consistency condition(ZCR)

$$\bar{\partial}U - \partial V + [U, V] = 0$$

  gives the reduced equation
- has been done with multiple sigma models, including $O(3)$

4.6 Lax pair reduction of $O(3)$

For $CP(N) = SU(N+1)/U(1)$ and thus $O(3) \cong CP(1)$, this is done by taking

$$U = \lambda A + C$$
$$V = \Lambda^{-1} B + D$$

$A, B \in \mathfrak{p}$ are elements of the orthogonal complement of the lie algebra $\mathfrak{g} \equiv \mathfrak{u}(N) \oplus \mathfrak{u}(1)$. with $\mathfrak{cp}(n) = \mathfrak{p} \oplus \mathfrak{g}$ reduce the system into the form

$$U = \zeta A + C$$
$$V = \zeta^{-1} B$$

4.6 Lax pair reduction of $O(3)$

This Reduction gives us the system

$$U = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \partial\varphi & 0 \\ 0 & -\partial\varphi \end{pmatrix} \quad V = \lambda^{-1} \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix}$$

which is exactly the sine-Gordon system. The initial Lax system can be annalized.

4.7 Why go through all this?

- Find an exactly integrable solution
- Study similarity to Liouville: Liouville $\rightarrow$ particle in exponential field
- Finding exact S-matrix in QFT

Questions

4.8 Appendix 1

The energy-momentum components for $O(3)$ are

$$T_{++} = 4\frac{\partial u \partial u^*}{(1+uu^*)^2} \Rightarrow \bar{\partial} T_{++} = 0$$

$$T_{--} = 4\frac{\bar{\partial} u \bar{\partial} u^*}{(1+uu^*)^2} \Rightarrow \partial T_{--} = 0$$

$$T_{+-} = T_{-+} = 0.$$

coordinate transformation invariance is preserved so we can set

$$T_{++} = T_{--} = 1$$

assign $\rho \equiv 1 + uu^*$ and write

$$4\partial u \partial u^* = 4\bar{\partial} u \bar{\partial} u^* = \rho^2$$

4.8 Appendix 1

introduce the field φ as

$$e^{i\varphi} = 4\frac{\bar{\partial}u\partial u^*}{\rho^2}, \quad e^{-i\varphi} = 4\frac{\partial u\bar{\partial}u^*}{\rho^2}$$

$$e^{i\varphi}e^{-i\varphi} = 1, \quad e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi = \mathcal{L}$$

By substituting ρ^2 from the stress-energy conditions we get

$$e^{i\varphi} = \frac{\bar{\partial}u}{\partial u} = \frac{\partial u^*}{\bar{\partial}u^*}$$

$$e^{-i\varphi} = \frac{\bar{\partial}u^*}{\partial u^*} = \frac{\partial u}{\bar{\partial}u}$$

After taking the logarithm we get the expression for phi and study the dynamics:

4.8 Appendix 1

$$\varphi = \ln \bar{\partial} u - \ln \partial u$$

$$\partial \bar{\partial} \varphi = \bar{\partial} \partial \ln \bar{\partial} u - \partial \bar{\partial} \ln \partial u$$

To find the derivatives of the logarithms rewrite the equations of motion

$$\bar{\partial}(\ln \partial u) = \frac{2u^*}{\rho} \bar{\partial} u$$

$$\partial\left(\ln \bar{\partial} u\right) = \frac{2u^*}{\rho} \partial u$$

4.8 Appendix 1

After differentiating the terms and simplifying

$$\bar{\partial}\partial \ln \bar{\partial} u = 2\bar{\partial} u \frac{\partial u^* - u u^* \partial u}{\rho^2}$$

$$\partial\bar{\partial} \ln \partial u = 2\partial u \frac{\bar{\partial} u^* - u u^* \bar{\partial} u}{\rho^2}$$

substitute back into the equation for $\partial\bar{\partial}\varphi$ and simplify

$$\partial\bar{\partial}\varphi = -2\frac{\partial u \bar{\partial} u^* - \bar{\partial} u \partial u^*}{\rho^2} = -\frac{e^{i\varphi} - e^{-i\varphi}}{2} = -\sin\varphi$$

4.9 Appendix 2

In lax pair reduction

$$A = \begin{pmatrix} 0 & \varphi^\dagger \\ -\varphi & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & \psi^\dagger \\ -\psi & 0 \end{pmatrix} \quad C = \begin{pmatrix} -\operatorname{tr}(X) & 0 \\ 0 & X \end{pmatrix}$$

with φ and ψ vectors, $X^\dagger = -X$ and D is analogous to C

- require $\varphi\varphi^\dagger = \psi\psi^\dagger = 1$
- choose a moving orthonormal frame[1]
- gauge transforming the matrices

$$\Longrightarrow U = \lambda A + C$$
$$V = \lambda^{-1} B$$

[1] in our case $g = (Z, Y)$, where Z is the physical field of the model and Y is a complex matrix

4.9 Appendix 2

$$A = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & \psi \\ -\psi & 0 \end{pmatrix} \quad C = \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & c_2 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & c_N \\ 0 & -c_2 & \cdots & -c_n & -c_1 \end{pmatrix}$$

where

$$c_1 = \frac{\psi_{N+1}\partial\psi_{N+1} + \sum_{i=2}^{N} \partial\psi_i \cdot \bar{\psi}_i}{3|\psi_{N+1}|^2 - 1}$$

$$c_i = \frac{\partial\psi_i}{\psi_{N+1}} + \frac{\psi_i}{\psi_{N+1}} c_1$$

for general $CP(N)$

4.9 Appendix 2

For $N = 1$ we get the result for the $\mathbb{C}P^1$ model

$$U = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} c_1 & 0 \\ 0 & -c_1 \end{pmatrix} \quad V = \lambda^{-1} \begin{pmatrix} 0 & \overline{\psi_2} \\ -\psi_2 & 0 \end{pmatrix}$$

where $c_1 = \frac{\psi_2 \partial \bar{\psi}_2}{2|\psi_2|^2}$. take $\psi_2 = e^{-i\varphi} \Rightarrow c_1 = \frac{i}{2}\partial\varphi$

the system becomes

$$U = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \partial\varphi & 0 \\ 0 & -\partial\varphi \end{pmatrix} \quad V = \lambda^{-1} \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix}$$

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