

Integrability and Scattering in Witten's Black Hole Model

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The $SL(2, \mathbb{R})/U(1)$ WZW theory

We study the scattering problem in $SL(2, \mathbb{R})/U(1)$ WZW theory.

This theory is also called Witten's black hole model, or simply, 'Witten's cigar', since the target space in certain polar coordinates looks like a cigar.

The model is integrable similarly to Liouville theory. In particular, the general solution of the equations of motion can be expressed in terms of the chiral (and anti-chiral) functions in a compact analytical form.

The form of the general solution allows to analyze the behavior of the interacting field at the time asymptotics, which shows that the interacting field of the theory asymptotically becomes a free massless field.

This free field itself has the chiral structure and the chiral fields which parameterize the general solution are expressed in terms of the asymptotic (*in* or *out*) fields.

As a result the interacting field is expressed by the asymptotic field.

Asymptotic fields and symmetries

One then finds explicit relations between the *in* and *out* fields as a non-trivial canonical transformation and the corresponding generating functional F defines the semiclassical S -matrix.

The $SL(2, \mathbb{R})/U(1)$ WZW theory is a sigma-model and it naturally has the 2d conformal symmetry generated by the stress tensor components $(T, \bar{T})$.

In addition to these chiral currents, there are also the W -currents $(W, \bar{W})$, which define higher symmetries and provide complete integrability.

The chiral currents have a free-field form in terms of the asymptotic fields and due to the chiral structure their form is the same for *in* and *out* fields.

This leads to an infinite set of equations for the generating functional F . These equations describe the symmetry of the theory.

The S -matrix

In quantum theory these equations are reformulated for the S -matrix, using the free-field representation of the quantum currents.

By these equations one finds low level transition amplitudes in a closed form. They are rational functions of $\hbar$ and the momentum zero modes. However, generalizations to arbitrary levels become problematic.

An alternative scheme could be a functional integral representation for the S -matrix, which effectively works for Liouville theory.

(G.J., Theisen, 2021)

This scheme is based on a Legendre transformation which relates the generating function F to a new functional A constructed in closed form.

In this talk we present a generalization of this scheme for Witten's cigar.

We describe the $SL(2, \mathbb{R})/U(1)$ WZW theory by the action

$$S = \frac{1}{2} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \frac{(\partial_\tau \Phi_1)^2 + (\partial_\tau \Phi_2)^2 - (\partial_\sigma \Phi_1)^2 - (\partial_\sigma \Phi_2)^2}{1 + e^{2\Phi_1}}$$

where (σ, τ) are the standard spacetime coordinates on the cylinder and (Φ_1, Φ_2) are scalar fields.

This action corresponds to a sigma-model and it is convenient to use the chiral coordinates $(x = \tau + \sigma, \bar{x} = \tau - \sigma)$.

The equation of motion for the complex field $\Phi = \Phi_1 + i\Phi_2$ reads

$$\partial_{\bar{x}} \partial_x \Phi = \frac{e^{\Phi^* + \Phi}}{1 + e^{\Phi^* + \Phi}} \partial_{\bar{x}} \Phi \partial_x \Phi$$

The equation of motion provides the chirality conditions

$$\partial_{\bar{x}}T = 0 \quad \partial_x\bar{T} = 0 \quad \partial_{\bar{x}}U = 0 \quad \partial_x\bar{U} = 0$$

where $(T, \bar{T})$ are the standard components of the stress tensor in this sigma-model

$$T = \frac{\partial_x\Phi^* \partial_x\Phi}{1 + e^{\Phi^* + \Phi}} \quad \bar{T} = \frac{\partial_{\bar{x}}\Phi^* \partial_{\bar{x}}\Phi}{1 + e^{\Phi^* + \Phi}}$$

and $(U, \bar{U})$ have the following form

$$U = \frac{\partial_x^2\Phi}{\partial_x\Phi} - \partial_x\Phi + \frac{\partial_x\Phi^*}{1 + e^{\Phi^* + \Phi}} \quad \bar{U} = \frac{\partial_{\bar{x}}^2\Phi}{\partial_{\bar{x}}\Phi} - \partial_{\bar{x}}\Phi + \frac{\partial_{\bar{x}}\Phi^*}{1 + e^{\Phi^* + \Phi}}$$

Integrability

The field $V = e^{-\Phi}$ satisfies the linear equations

$$\partial_x^2 V = U(x) \partial_x V + T(x) V \qquad \partial_{\bar{x}}^2 V = \bar{U}(\bar{x}) \partial_{\bar{x}} V + \bar{T}(\bar{x}) V$$

The Lagrangian takes a more familiar form in terms of the field V

$$\mathcal{L} = \frac{1}{4\pi} \frac{\partial_{\bar{x}} V^* \partial_x V + \partial_x V^* \partial_{\bar{x}} V}{1 + V^* V}$$

From the linear equations follows that the field $V(x, \bar{x})$ can be written as

$$V(x, \bar{x}) = \psi(x) \bar{\psi}(\bar{x}) + \tilde{\psi}(x) \tilde{\bar{\psi}}(\bar{x})$$

where $(\psi(x), \tilde{\psi}(x))$ and $(\bar{\psi}(\bar{x}), \tilde{\bar{\psi}}(\bar{x}))$ are linearly independent solutions of the first and the second linear equations, respectively, i.e.

$$\Psi''(x) = U(x) \Psi'(x) + T(x) \Psi(x) \qquad \tilde{\Psi}''(x) = U(x) \tilde{\Psi}'(x) + T(x) \tilde{\Psi}(x)$$

The general solution

In this way one gets the general solution of the equations of motion, which splits in different monodromy classes (Muller, Weigt, 1997) .

Scatterings are described by the hyperbolic monodromy, where the parameterizing chiral fields are the free-field exponentials

$$\psi(x) = e^{-\phi_1(x) - i\phi_2(x)} \quad \tilde{\psi}(x) = e^{-\tilde{\phi}_1(x) - i\tilde{\phi}_2(x)}$$

One can show that they are related by

$$\psi^*(x) \tilde{\psi}(x) - \tilde{\psi}^*(x) \psi(x) = i$$

The free fields obey the standard mode expansions

$$\begin{aligned} \phi_1(x) &= q_1 + \frac{1}{2}p_1x + i \sum_{n \neq 0} \frac{a_n}{n} e^{-inx} & \tilde{\phi}_1(x) &= \tilde{q}_1 + \frac{1}{2}\tilde{p}_1x + i \sum_{n \neq 0} \frac{b_n}{n} e^{-inx} \\ \phi_2(x) &= q_2 + \frac{1}{2}p_2x + i \sum_{n \neq 0} \frac{c_n}{n} e^{-inx} & \tilde{\phi}_2(x) &= \tilde{q}_2 + \frac{1}{2}\tilde{p}_2x + i \sum_{n \neq 0} \frac{d_n}{n} e^{-inx} \end{aligned}$$

where $\tilde{p}_1 = -p_1 < 0$, $\tilde{p}_2 = p_2 + 2N$, with integer N , which describes a winding number. We consider the case $N = 0$.

T and W

The field $e^{2\Phi_1}$ vanishes at the time asymptotics.

Therefore, at $\tau \rightarrow \mp\infty$, Φ_1 and Φ_2 become free massless fields.

The form of the chiral fields $T(x)$ and $U(x)$ is the same in terms of the *in* and *out* fields.

To simplify the analysis of the chiral fields we introduce the W -current

$$W(x) = \frac{i}{4} (U(x) - U^*(x)) T(x)$$

and the conditions $T_{\text{in}}(x) = T_{\text{out}}(x)$ and $W_{\text{in}}(x) = W_{\text{out}}(x)$ become

$$\phi_1'^2 + \phi_2'^2 = \tilde{\phi}_1'^2 + \tilde{\phi}_2'^2$$

$$\phi_1'^2 \phi_2' + \phi_2'^3 + \frac{1}{2} \phi_1'' \phi_2' - \frac{1}{2} \phi_1' \phi_2'' = \tilde{\phi}_1'^2 \tilde{\phi}_2' + \tilde{\phi}_2'^3 + \frac{1}{2} \tilde{\phi}_1'' \tilde{\phi}_2' - \frac{1}{2} \tilde{\phi}_1' \tilde{\phi}_2''$$

Symplectic structure

The Poisson brackets of the Fourier modes are defined by the symplectic form of the chiral fields, which are similar for all four fields and they read

$$\omega = \delta p_1 \wedge \delta q_1 + 2i \sum_{m>0} \frac{\delta a_m^* \wedge \delta a_m}{m}$$

Since the asymptotic fields are related to each other by a canonical transformation, we can introduce the generating functional of this transformation defined by

$$\delta F = (\tilde{q}_1 + q_1) \delta p_1 - (\tilde{q}_2 - q_2) \delta p_2 - 2i \sum_{m>0} \frac{1}{m} (b_m \delta b_m^* + d_m \delta d_m^* + a_m^* \delta a_m + c_m^* \delta c_m)$$

where the 1-form on the right hand side is the difference between the presymplectic forms of the *out* and the *in* fields.

Equations for F

Thus, F is treated as a function of the modes $(p_1, p_2, b_m^*, d_m^*, a_m, c_m)$, with $m > 0$, and its partial derivatives satisfy the relations

$$\begin{aligned}\frac{\partial F}{\partial p_1} &= \tilde{q}_1 + q_1 & \frac{\partial F}{\partial p_2} &= q_2 - \tilde{a}_2 \\ \frac{\partial F}{\partial b_m^*} &= -\frac{2i}{m} b_m & \frac{\partial F}{\partial d_m^*} &= -\frac{2i}{m} d_m \\ \frac{\partial F}{\partial a_m} &= -\frac{2i}{m} a_m^* & \frac{\partial F}{\partial c_m} &= -\frac{2i}{m} c_m^*\end{aligned}$$

Our goal is to describe the functional F .

It provides a semiclassical S -matrix.

New chiral fields

We introduce the chiral fields

$$\varphi(x) = \phi_1(x) + \tilde{\phi}_1(x) = \sum_{n \in \mathbb{Z}} \varphi_n e^{-inx}$$

$$\chi(x) = \phi_2(x) + \tilde{\phi}_2(x) = p_2 x + \sum_{n \in \mathbb{Z}} \chi_n e^{-inx}$$

$$j(x) = p_1 + 2a'(x) - 2b^{*'}(x) = \sum_{n \in \mathbb{Z}} j_n e^{-inx}$$

$$k(x) = 2c'(x) - 2d^{*'}(x) = \sum_{n \neq 0} k_n e^{-inx}$$

Here, $a(x)$ is the positive frequency part of $\phi_1(x)$, $b^*(x)$ is the negative frequency part of $\tilde{\phi}_1(x)$, etc.

Note that $\chi(x)$ is not periodic and $k(x)$ has no zero mode.

The relations between the chiral fields

The asymptotic chiral fields are related by

$$\frac{\phi'_1 - \tilde{\phi}'_1}{\phi'_2 + \tilde{\phi}'_2} = \frac{\tilde{\phi}'_2 - \phi'_2}{\phi'_1 + \tilde{\phi}'_1}$$

We denote the ratio here by $R(x)$ and obtain

$$\phi'_1 = \frac{1}{2}(\varphi' + R\chi') \quad \tilde{\phi}'_1 = \frac{1}{2}(\varphi' - R\chi') \quad \phi'_2 = \frac{1}{2}(\chi' - R\varphi') \quad \tilde{\phi}'_2 = \frac{1}{2}(\chi' + R\varphi')$$

which leads to the following equation for $R(x)$

$$R' - (R + R^3)\varphi' = 0$$

Its integration yields

$$R(x) = \frac{c e^{\varphi(x)}}{\sqrt{1 - c^2 e^{2\varphi(x)}}}$$

where c is an integration constant.

Solving the relations

Solving the relations between the chiral fields one gets

$$\phi'_1 - \tilde{\phi}'_1 = j - \varphi'_+ + \varphi'_- \quad \phi'_2 - \tilde{\phi}'_2 = k - \chi'_+ + \chi'_-$$

where φ_+ and φ_- are the positive and negative frequency parts of φ , respectively, and similarly for the field χ .

Then, we find

$$j = \varphi'_+ - \varphi'_- + \frac{c e^{\varphi(x)} \chi'}{\sqrt{1 - c^2 e^{2\varphi(x)}}} \quad k = \chi'_+ - \chi'_- - \frac{c e^{\varphi(x)} \varphi'}{\sqrt{1 - c^2 e^{2\varphi(x)}}}$$

The integration constant c is specified from the relation mentioned above

$$2 \sin(\phi_2 - \tilde{\phi}_2) = e^{\phi_1 + \tilde{\phi}_1}$$

and it provides $|c| = 1/2$.

Legendre transformation

Our goal is to construct a functional integral representation for the S -matrix, based on a Legendre transformation of the functional F .

For this purpose we rewrite the relations between the chiral fields as

$$j(x) = 2\pi \frac{\delta A[\varphi, \chi]}{\delta \varphi(x)} \qquad k_n = \frac{\partial A[\varphi, \chi]}{\partial \chi_{-n}}$$

where $A[\varphi, \check{\chi}]$ is the functional

$$A[\varphi, \check{\chi}] = -\frac{i}{2} \sum_{n \in \mathbb{Z}} |n| \varphi_{-n} \varphi_n - \frac{i}{2} \sum_{n \in \mathbb{Z}} |n| \chi_{-n} \chi_n + \int_0^{2\pi} \frac{dx}{2\pi} \arcsin[c e^{\varphi(x)}] \chi'(x)$$

Legendre transformation

We then introduce the Legendre transformation $A[\varphi, \check{\chi}] \mapsto \tilde{F}[j, k]$

$$\tilde{F}[j, k] = \sum_{n \in \mathbb{Z}} j_{-n} \varphi_n + \sum_{n \neq 0} k_{-n} \chi_n - A[\varphi, \check{\chi}]$$

This provides

$$\frac{\delta \tilde{F}[j, k]}{\delta j(x)} = \frac{1}{2\pi} \varphi(x) \qquad \frac{\partial \tilde{F}[j, k]}{\partial k_n} = \chi_{-n}$$

and, therefore, the functional

$$\tilde{F} + 2i \sum_{m>0} \frac{b_m^* a_m}{m} + 2i \sum_{m>0} \frac{d_m^* c_m}{m}$$

reproduces F up a term which is only p_2 dependent.

Functional integral

Since $\tilde{F}[j, k]$ and $A[\varphi, \tilde{\chi}]$ are related by the Legendre transformation, the saddle point calculation of the integral

$$\int d\varphi_0 \prod_{n \neq 0} d\varphi_n d\chi_n \exp \frac{i}{\hbar} \left(p_1 \varphi_0 + \sum_{n \neq 0} (j_{-n} \varphi_n + k_{-n} \chi_n) - A[\varphi, \tilde{\chi}] \right)$$

yields

$$e^{\frac{i}{\hbar} \tilde{F}}$$

This describes the non-zero mode transition at list semi-classically.

Note that the same scheme for Liouville theory gives a functional integral representation of the S -matrix which reproduces exact results.

(G.J., Theisen, 2021)

Summary

The quantum version of the Legendre transformation $A \mapsto \tilde{F}$ is treated as

$$e^{\frac{i}{\hbar} \tilde{F}[b^*, a]} = \int d\varphi^* d\varphi e^{\frac{i}{\hbar} (a\varphi^* - b^*\varphi - A[\varphi^*, \varphi])}$$

The saddle point calculation of the integral yields the classical relation.

As it was mentioned above, in Liouville theory this relation was checked in by various quantum calculations.

Low level transition amplitudes in $SL(2, \mathbb{R})/U(1)$ WZW theory are known in a closed form ([unpublished notes G.J., Theisen](#)).

This defines a part of the left hand side and it is natural to calculate the corresponding right hand side, investigating the functional integral for the constructed functional A .

Note that a similar scheme works for the $SL(3, \mathbb{R})$ Toda theory and $\mathcal{N} = 1$ supersymmetric Liouville theory.