Baryon Spectroscopy in Coupled Channels

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Hadron 2015, Sept. 17, 2015
The excited hadron spectrum: testing ground for theories of the strong force at low and medium energy

**Experimental study of hadronic reactions**

- $\gamma + p \to X$
- $\gamma + p \to p + \pi^+$
- $\gamma + p \to p + \pi^0$
- $\gamma + p \to p + \pi^0$
- $\gamma + p \to K^+ + \Lambda$
- $\gamma + p \to p + \eta$

source: ELSA; data: ELSA, JLab, MAMI

$\Rightarrow$ **Partial wave decomposition:**
- decompose data with respect to a conserved quantum number: $J^P$

**Theoretical predictions** of excited hadrons e.g. from lattice calculations:

$\Rightarrow$ **Missing resonance problem**

$m_\pi = 396$ MeV [Edwards et al., Phys.Rev. D84 (2011)]
Resonances

\( J^P = 1/2^+, \ I = 3/2 \)

Breit-Wigner parameterization:

\[
\mathcal{M}_{ba}^{\text{Res}} = \frac{-g_b g_a}{E^2 - M_{BW}^2 + iE \Gamma_{BW}}
\]

- \( M_{BW}, \Gamma_{BW} \) channel dependent
- background? overlapping resonances? thresholds?

Resonances: poles in the \( T \)-matrix

- Pole position \( E_0 \) is the same in all channels
- thresholds: branch points

\( \Re(E_0) = \) “mass”
\( -2\Im(E_0) = \) “width”
residues \( \rightarrow \) branching ratios

How to connect lattice eigenvalues to resonance poles?

But first: How to construct a suitable amplitude to determine resonance poles?

Points: SAID 2006 and CM12
The scattering matrix

**Unitarity:** \( SS^\dagger = 1 \iff -i(T - T^\dagger) = TT^\dagger \)

- **3-body unitarity:**
  - discontinuities from \( t \)-channel exchanges
  - \( \rightarrow \) Meson exchange from requirements of the \( S \)-matrix \ ([Aaron, Almado, Young, Phys. Rev. 174, 202 (1968)]

**Other cuts**

- to approximate left-hand cut \( \rightarrow \) Baryon \( u \)-channel exchange
- \( \sigma, \rho \) exchanges from crossing plus analytic continuation.

\[ \vec{q} = \vec{p}_1 - \vec{p}_3 \]
\[ \vec{q} = \vec{p}_1 - \vec{p}_4 \]
\[ \vec{q} = \vec{p}_1 + \vec{p}_2 = 0 \]
Pion- and photon-induced reactions
Connections to QCD

Properties of the amplitude
Results
Connected channels
Pion- and photon-induced reactions
Connections to QCD

Results

Properties of the amplitude

3-body
Subthreshold cuts

CC

LHC

SNC

πN

ππN

Physical

ηN

KΛ

KΣ

ρN

σN

πΔ
Relevance of three-body dynamics

- Roper pole + $\pi\Delta$ branch point $\rightarrow$ non-standard resonance shape.
- See results by GWU/SAID data analysis center.

Inclusion of full analytic structure important to avoid false pole signals in baryon spectroscopy.

Where is the $3^* N(1710)$?
[S. Ceci, M.D. et al, PRC84, 2011]

Fit of a model without $\rho N$ branch point (CMB type) [solid lines] to the Jülich amplitude [dashed lines]

CMB fit to JM has pole at $1698 - 130 i$ MeV, simulates missing branch point.
A dynamical coupled-channel approach: the hadronic Jülich model

Dynamical coupled-channels (DCC): **simultaneous** analysis of different reactions

The scattering equation in partial wave basis

\[
\langle L' S' p' | T_{\mu\nu}^{IJ} | L S p \rangle = \langle L' S' p' | V_{\mu\nu}^{IJ} | L S p \rangle + \sum_{\gamma, L'' S''}^{\infty} \int_{0}^{\infty} q^2 dq \langle L' S' p' | V_{\mu\gamma}^{IJ} | L'' S'' q \rangle \frac{1}{E - E_{\gamma}(q) + i\epsilon} \langle L'' S'' q | T_{\gamma\nu}^{IJ} | L S p \rangle
\]

- **t- and u-channel:** \( T^{NP} \)
dynamical generation of poles

- partial waves strongly correlated

- potentials \( V \) constructed from effective \( \mathcal{L} \)

- bare s-channel resonance states; Meaning? → Talk by W. Schweiger
A dynamical coupled-channel approach: the hadronic Jülich model

**Dynamical coupled-channels (DCC):** simultaneous analysis of different reactions

### The scattering equation in partial wave basis

\[
\langle L'S'p'| T^{jj}_{\mu\nu} | LSp \rangle = \langle L'S'p'| V^{jj}_{\mu\nu} | LSp \rangle + \sum_{\gamma, L''S''} \int_0^\infty q^2 dq \langle L'S'p'| V^{jj}_{\mu\gamma} | L''S''q \rangle \frac{1}{E - E_\gamma(q) + i\epsilon} \langle L''S''q| T^{jj}_{\gamma\nu} | LSp \rangle
\]

- **Analyticity** is respected (correct structure of branch points and cuts)  
  \(\leftrightarrow\) reliable extraction of resonance parameters
- **Unitarity**
- \(J \leq \frac{9}{2}\)

\[
\begin{align*}
\pi N &\quad \pi\pi N \\
1077 &\quad 1215
\end{align*}
\]

\[
\begin{align*}
\eta N &\quad K\Lambda &\quad K\Sigma \\
1486 &\quad 1611 &\quad 1688
\end{align*}
\]

\[
\begin{align*}
\pi\Delta &\quad \sigma N &\quad N\rho \\
1486 &\quad 1611 &\quad 1688
\end{align*}
\]

\(\sim 2.3\ \text{GeV}\)
\[ \pi^- p \rightarrow \eta N \]

\[ \pi^- p \rightarrow K^0 \Lambda \]


\[ \pi^- N \rightarrow \eta N, K \Lambda \] selected results

\[ \pi^- p \rightarrow \eta N \]

\[ \pi^- p \rightarrow K^0 \Lambda \]

Full results
**πN → KΣ**

**Selected results**

\(\pi^- p \rightarrow K^0 \Sigma^0\)

\(\pi^- p \rightarrow K^+ \Sigma^-\)

\(\pi^+ p \rightarrow K^+ \Sigma^+\)

\(E = 1694\text{ MeV}\)  
\(E = 2026\text{ MeV}\)

\(1763\text{ MeV}\)  
\(2305\text{ MeV}\)

\(z = 1729\text{ MeV}\)  
\(2261\text{ MeV}\)

No polarization data!

\(\beta [\text{rad}]\)

\(2031\text{ MeV}\)  
\(2021\text{ MeV}\)

**Full results**

*Physics Opportunities with Hadron Beams, arXiv:1503.07763*
Improvement in Modern Experimental Facilities: $\pi N \rightarrow \pi N$

EPECUR & GWU/SAID, Alekseev et al., PRC91, 2015

Black: WI08 prediction; Red: WI14 fit; green: KA84.
### Data base of $\gamma p \rightarrow \pi^0 p, \pi^+ n$

D. Rönchen, M.D. et al., EPJA50, 2014

<table>
<thead>
<tr>
<th>Line style</th>
<th>Fit 1</th>
<th>Fit 2</th>
</tr>
</thead>
<tbody>
<tr>
<td># of data</td>
<td>21,627</td>
<td>23,518</td>
</tr>
<tr>
<td>Excluded data</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi^0 p$: $E &gt; 2.33$ GeV and $\theta &lt; 40^\circ$ for $E &gt; 2.05$ GeV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi^+ n$: $E &gt; 2.26$ GeV and $\theta &lt; 9^\circ$ for $E &gt; 1.60$ GeV</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| $ds/d\Omega, P, T$ | included | included |
| $\Sigma$ | included | included |
| (CLAS 2013 predicted) | | |
| $\Delta \sigma_{31}, G, H$ | predicted | included |
| $E, F, C_{x'L}, C_{z'L}$ | predicted | predicted |

Minimization using parallelized code on JUROPA (grant from Jülich Supercomputing Centre)

**Complete experiment**

one possibility:

$\sigma, \Sigma, T, P, E, G, C_x, C_z$

$\Rightarrow$ Influence of new double polarization observables on resonance parameters
Single polarization observables

- $d\sigma/d\omega$
  - $\gamma p \rightarrow \pi^0 p$
  - $\gamma p \rightarrow \pi^+ n$

- Beam asymmetry $\Sigma$
  - $\gamma p \rightarrow \pi^0 p$
  - $\gamma p \rightarrow \pi^+ n$

Polarization:
- Beam
- Target
- Recoil

**Results**


Double polarization observables

**$G$**

\[
\gamma p \rightarrow \pi^0 p
\]

\[
\begin{array}{c|c|c|c|c}
0 & 1543 & 1603 & 1660 & 1716 \\
30 & 0 & 0 & 0 & 0 \\
60 & 0 & 0 & 0 & 0 \\
90 & 0 & 0 & 0 & 0 \\
120 & 0 & 0 & 0 & 0 \\
150 & 0 & 0 & 0 & 0 \\
180 & 0 & 0 & 0 & 0 \\
\end{array}
\]

**$C_{x'}$**

\[
\gamma p \rightarrow \pi^0 p
\]

\[
\begin{array}{c|c|c|c|c}
\theta \text{ [deg]} & 0 & 75 & 105 & 140 \\
\end{array}
\]

**Predictions**

- **fit 1, angle averaged**
- **fit 2, angle averaged**
- **fit 2, not angle averaged**

[1] Sikora et al. 2013 (MAMI)
[3] Luo et al. 2012 (JLab)
### Photocouplings at the pole

Selected results

\[ \tilde{A}_h^{\text{pole}} = A_h^{\text{pole}} e^{i\vartheta_h} \]

\[ h = 1/2, \ 3/2 \]

\[ \tilde{A}_h^{\text{pole}} = I_F \sqrt{\frac{q_p}{k_p} \frac{2\pi (2J+1)E_0}{m_N r_{\pi N}}} \text{Res} \ A_{L \pm}^h \]

- \( I_F \): isospin factor
- \( q_p (k_p) \): meson (photon) momentum at the pole
- \( J = L \pm 1/2 \): total angular momentum
- \( E_0 \): pole position
- \( r_{\pi N} \): elastic \( \pi N \) residue

<table>
<thead>
<tr>
<th></th>
<th>( A_{1/2}^{\text{pole}} )</th>
<th>( \vartheta^{1/2} )</th>
<th>( A_{3/2}^{\text{pole}} )</th>
<th>( \vartheta^{3/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>([10^{-3} \text{ GeV}^{-1/2}])</td>
<td>([\text{deg}])</td>
<td>([10^{-3} \text{ GeV}^{-1/2}])</td>
<td>([\text{deg}])</td>
</tr>
<tr>
<td>( \mathcal{N}(1710) \ 1/2^+ )</td>
<td>15 \hspace{1cm} 28^{+9}_{-2}</td>
<td>13 \hspace{1cm} 77^{+20}_{-9}</td>
<td>1 \hspace{1cm} 229^{+3}_{-4}</td>
<td>15 \hspace{1cm} 15^{+0.3}_{-0.4}</td>
</tr>
<tr>
<td>( \Delta(1232) \ 3/2^+ )</td>
<td>-116 \hspace{1cm} -114^{+10}_{-3}</td>
<td>-27 \hspace{1cm} -27^{+4}_{-2}</td>
<td>-231 \hspace{1cm} -229^{+3}_{-4}</td>
<td>-15 \hspace{1cm} -15^{+0.3}_{-0.4}</td>
</tr>
</tbody>
</table>

**Fit 1**: only **single** polarization observables included

**Fit 2**: also **double** polarization observables included
First FROST results: $E$ in $\vec{\gamma}p \rightarrow \pi^+ n$

PLB 750 (2015); data not preliminary any more.

Data: CLAS (USC/Strauch et al.); preliminary
SAID analysis (prediction and re-analysis)
Jülich Athens Washington (prediction and re-analysis)

→ Significant changes of helicity amplitudes $A^{1/2}$, $A^{3/2}$
Impact on resonance parameters

- $A [10^{-3} \text{ GeV}^{-1/2}]$; $\vartheta [\text{deg}]$.
- Pole values (P) or Breit-Wigner Parameters (BW).
- ";" separates before and after new FROST $E$-measurement.

<table>
<thead>
<tr>
<th></th>
<th>$A^{1/2}_{\text{pole}}$</th>
<th>$\vartheta^{1/2}$</th>
<th>$A^{3/2}_{\text{pole}}$</th>
<th>$\vartheta^{3/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(1440) 1/2^+$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SAID BW</td>
<td></td>
<td>-60 $\pm$ 5; -56 $\pm$ 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jülich P</td>
<td></td>
<td>-52; -54</td>
<td>-51; -43</td>
<td></td>
</tr>
<tr>
<td>BnGa P</td>
<td>-44 $\pm$ 7</td>
<td>-38 $\pm$ 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ANL-Osaka P</td>
<td>49</td>
<td>-10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>WTS P</td>
<td>-66 $\pm$ 5</td>
<td>-38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PDG BW</td>
<td>-60 $\pm$ 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1675) 5/2^-$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SAID BW</td>
<td></td>
<td>10 $\pm$ 3; 13 $\pm$ 1</td>
<td>16 $\pm$ 4; 16 $\pm$ 1</td>
<td></td>
</tr>
<tr>
<td>Jülich P</td>
<td>28; 22</td>
<td>40; 38</td>
<td>63; 36</td>
<td>-19; -41</td>
</tr>
<tr>
<td>BnGa P</td>
<td>24 $\pm$ 3</td>
<td>-16 $\pm$ 5</td>
<td>26 $\pm$ 8</td>
<td>-19 $\pm$ 6</td>
</tr>
<tr>
<td>ANL-Osaka P</td>
<td>5</td>
<td>-22</td>
<td>33</td>
<td>-23</td>
</tr>
<tr>
<td>PDG BW</td>
<td>19 $\pm$ 8</td>
<td></td>
<td>15 $\pm$ 9</td>
<td></td>
</tr>
</tbody>
</table>
Fit results $\gamma p \rightarrow \eta p$

- **Differential cross section**

- **Beam asymmetry**

- **Recoil polarization**
$T$ and $F$ in $\gamma p \rightarrow \eta p$ (MAMI)

Data: Akondi et al. (A2 at MAMI) *PRL* 113, 102001 (2014)

![Graph showing $T$ and $F$](image)

<table>
<thead>
<tr>
<th>Beam</th>
<th>Target</th>
<th>Recoil</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$+y$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$-y$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Beam</th>
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<th>Recoil</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+1$</td>
<td>$+x$</td>
<td>0</td>
</tr>
<tr>
<td>$-1$</td>
<td>$+x$</td>
<td>0</td>
</tr>
</tbody>
</table>
FROST $E$ in $\gamma p \rightarrow \eta p$

arXiv: 1507.00325
FROST $E$ in $\gamma p \rightarrow \eta p$: Excitation function

arXiv: 1507.00325

$E$ vs $W$ [MeV] for different values of $\cos \theta$.
Structure at $W = 1.68$ GeV: Conventional Resonances

NO narrow state at $W = 1.68$ GeV needed.
Partial convergence of Partial Wave Analyses
Before-after scenario for Jülich/Bonn, SAID, Bonn-Gatchina

Change in overall variance after including latest $\pi^0 p$ (and other) polarization measurements from ELSA (M. Gottschall, J. Hartmann, A. Thiel, 2015), MAMI (Hornidge 2015), JLab (Dugger, Luo, 2015).
Partial convergence of Partial Wave Analyses

Pairwise variances: (a) before, (b): after, (c): difference
Some questions for baryon resonance analysis

- How well does a certain observable constrain resonance properties? (taking into account data accuracy, and measured energy/angular range)
- How well would the measurement of a new observable further constrain and disentangle (correlations!) resonance properties?
- Are polarization measurements useful, despite the fact that much less accurate than cross section measurements?

Can these questions be addressed quantitatively?

- Uncertainties in EM resonance couplings → entire covariance matrix to get full & meaningful picture,

\[ \Sigma_{ij} = \frac{\partial^2 \chi^2}{\partial A_i \partial A_j} \]

- \( A_i \): coefficient of Laurent expansion around resonance pole (helicity coupling, modulus and phase, ...)
- \( \chi^2 \): the chi-square of a certain data set, e.g., \( E \) in \( \gamma p \rightarrow \eta p \), or: the chi-square of entire class of final states (is \( \eta \) photoproduction needed, compared to pion photoproduction?), or: simulated data with realistic errors, angular, and energy coverage (e.g.: would \( C_X \) in \( \gamma p \rightarrow \eta p \) be a good observable to disentangle resonance properties?)
Example: correlation matrix for $E$ in $\gamma p \rightarrow \eta p$

based on discussed FROST data

$$
\begin{align*}
&|A^{1/2}| N(1535)1/2^- \\
&|A^{1/2}| N(1650)1/2^- \\
&|A^{1/2}| N(1710)1/2^+ \\
&|A^{1/2}| N(1720)3/2^+ \\
&|A^{3/2}| N(1720)3/2^+ \\
&|A^{1/2}| N(1520)3/2^- \\
&|A^{3/2}| N(1520)3/2^- \\
&|A^{1/2}| N(1675)5/2^- \\
&|A^{3/2}| N(1675)5/2^- \\
&|A^{1/2}| N(1680)5/2^+ \\
&|A^{3/2}| N(1680)5/2^+
\end{align*}
$$

$|\rho_{\text{red}}|$

$|\rho_{\text{red}}|$
Figures of Merit (world data in $\gamma p \rightarrow \eta p$)

- Multicollinearity (resonance entanglement): $|\rho|$
  
  Comparison (smaller is better):
  
  $E : -\log |\rho| = 7.3,$
  $F : -\log |\rho| = 10.1,$
  $T : -\log |\rho| = 12,$
  $\Sigma : -\log |\rho| = 13,$
  $d\sigma/d\Omega : -\log |\rho| = 11,$
  all: $-\log |\rho| = 6.6,$
  $C_x : -\log |\rho| = 8.8$ (simulated).

- Overall quality: differential entropy (volume of the hyper-error-ellipse) $|\Sigma|$
  
  Comparison (smaller is better)
  
  $E : \log[|\Sigma|] = 98,$
  $F : \log[|\Sigma|] = 88,$
  
  ...$d\sigma/d\Omega : \log[|\Sigma|] = 11,$
  all: $\log[|\Sigma|] = 0.015,$
  $C_x : \log[|\Sigma|] = 62$ (simulated).

- Generalized variance, ...
Resonances decaying on the lattice

Eigenvalues in the finite volume

Avoided level crossing (resonance OR threshold)

Bound state: $E(L) \sim M_B + a e^{-bL}$
$J^P = 1/2^-$ in the finite volume

[M.D., Mai, Meißner, EPJA (2013)]

- Unitarized chiral interaction with NLO contact terms
Chiral extrapolation to a QCDSF lattice setup
No one-to-one mapping of levels to resonances → coupled channel analysis; hidden poles appear. Tomorrow: R. Molina on the Λ(1405) in the finite volume and comparison to recent lattice data.
Summary and Outlook

- Amplitude construction along general properties of the scattering amplitude
- Analysis of $\pi N \rightarrow \pi N, \eta N, K\Lambda$ and $K\Sigma$; $\gamma N \rightarrow \pi N, \eta N$.
- High-quality polarization observables from FROST@JLab start appearing; large impact on resonance spectrum and properties.
- Three-body effects cannot be ignored.
- Partial convergence between different PWAs observed as data become more complete.
- Correlations between electromagnetic resonance couplings can quantitatively address questions:
  - How well does a certain observable constrain resonance properties? (taking into account data accuracy, and measured energy/angular range)
  - How well would the measurement of a new observable further constrain and disentangle (!) resonance properties?
  - Are polarization measurements useful, despite the fact that much less accurate than cross section measurements? (Yes.)
- Connections to ab-initio Lattice QCD through extensions of the Lüscher formalism.
\( \pi N \rightarrow \pi N \) partial wave amplitudes

Coupled-channels at work

- Notation: \( L_{2/2J} \)
- Input to fit: energy-dependent partial wave analysis, GWU/SAID 2006 up to \( J = 9/2 \) (\( \sim H_{39} \))
\( \pi N \rightarrow \pi N \) partial wave amplitudes

Coupled-channels at work

Genuine \( N(1710) \frac{1}{2}^+ \):

- Black line: input to fit (energy-dependent solution)
- Inclusion of \( P_{11}(1710) \) necessary to improve \( K \Lambda \)
- Coupled-channels essential (Fit to \( \pi N \) observables)
**πN → πN partial wave amplitudes**

**Coupled-channels at work**

- **Black line**: input to fit (energy-dependent solution)
- **Inclusion of** $P_{11}(1710)$ **necessary to improve** $K\Lambda$
- **Coupled-channels** essential (Fit to $\pi N$ observables)

- Genuine $N(1710)_{1/2}^+$:

- Photocouplings of resonances
- high precision data from ELSA, MAMI, JLab... → resolve questionable/find new states

Photoproduction amplitude of pseudoscalar mesons:

\[
\hat{M} = F_1 \vec{\sigma} \cdot \vec{\epsilon} + iF_2 \vec{\epsilon} \cdot (\hat{k} \times \hat{q}) + F_3 \vec{\sigma} \cdot \hat{k} \hat{q} \cdot \vec{\epsilon} + F_4 \vec{\sigma} \cdot \hat{q} \hat{q} \cdot \vec{\epsilon}
\]

- \(\vec{q}\): meson momentum
- \(\hat{k} (\vec{\epsilon})\): photon momentum (polarization)
- \(F_i\): complex functions of the scattering angle

⇒ 16 polarization observables:
   - asymmetries composed of beam, target and/or recoil polarization measurements

⇒ 8 carefully selected observables: complete experiment
   ← Caveat: in reality more observables needed (data uncertainties)

Focus of the present analysis:

- extraction of electromagnetic resonance parameters
- flexible, “model-independent” parameterization of photo excitation

⇒ intermediate step towards a combined DCC analysis of pion- and photon-induced reactions

- Advantage: easy to implement, analyze large amounts of data
- Disadvantage: no information on microscopic reaction dynamics
Double polarization observables

\[ \Delta \sigma_{31} = -2 \frac{d\sigma}{d\Omega} E \]

- **\( \Delta \sigma_{31} \)**

\[ \gamma p \rightarrow \pi^0 p \]

\[ \gamma p \rightarrow \pi^+ n \]

Predictions of \( E \)

\[ \gamma p \rightarrow \pi^0 p \]


Challenges in Baryon Analysis

- Set common ground for different analyses - definite answers needed!
- Better $\pi N \rightarrow \pi N$, $\pi \pi N \eta N$, $KY$ data to improve analysis of photo- and electroproduction.
- Consistent data $\rightarrow$ Get rid of weighting data sets in fits (otherwise all statistical meaning lost).
- GWU/SAID: Provide correlations so other group can carry out statistically meaningful analyses.
- Answer the question: How sensitive are helicity couplings to the new set of $XY$ data? Maybe: Eigenvectors ordered by eigenvalues of the inverse correlation matrix

$$\frac{1}{2} \left( \frac{\partial^2 \chi_{XY}^2}{\partial A_i \partial A_j} \right)$$

$A_i$: $i$-th helicity coupling $|A|$ or complex phase $\theta$ ($A = |A| \exp(i\theta)$) at the pole.

- Approaching the precision frontier in baryon analysis: Joint effort by GWU/INS SAID, Jülich, JPAC.
Phenomenological challenges: bump in \( \gamma n \rightarrow \eta n \) (MAMI)

- \( \text{"N}(1685)\)" : large couplings to \( \gamma n, \eta n \)
- Pentaquark? [Prediction Polyakov et al.]
- Unusual combination of neutron helicity couplings?
  [BnGa]
- Coupled channel chiral dynamics from \( K\Lambda, K\Sigma \)? [M.D. et al.]

\[ \gamma d \rightarrow p n \eta \]
Jaegle et al., EPJA (2011)

- MAMI: Especially designed for neutral final states.
- Excellent energy resolution & angular coverage
- \( \pi^0 p \) final state needed for isospin decomposition in PWA; complementary to CLAS (\( \pi^+ n \)).
- Strong involvement of US physicists (GWU, KSU,...)
Phenomenological challenges: $K\Sigma$ photoproduction (ELSA)

$\gamma p \rightarrow K^0\Sigma^+$

- Sudden drop at $W \sim 2.05$ GeV
- Dynamic effect 
  [Ramos/Oset, PLB 2013]?
- CLAS, MAMI, ELSA: Strong complementary experimental program
- Joint data essential for partial wave analysis
- Still surprises in meson photoproduction.

CBELSA/TAPS [PLB 2012]
Analysis of pion-induced reactions

- calculate observables from $T$-matrix
- fit **free parameters** of $T$ to data or partial wave amplitudes

**s-channel: resonances** ($T^P$)

$$\sigma = \frac{1}{2} \frac{4\pi}{p^2} \sum_{JLS,L'S'} |T^{LL'S'}_{LS}|^2$$

with $\tau_{fi} = -\pi \sqrt{\rho_f \rho_i} T_{fi}$

$\rho$: phase factor

$t$- and $u$-channel exchange: "background" ($T^{NP}$)

m_{bare} + f_{\pi NN^*}

cut offs $\Lambda$ in form factors $\left( \frac{\Lambda^2 - m_{ex}^2}{\Lambda^2 + \vec{q}^2} \right)^n$

(couplings fixed from SU(3))

$\Rightarrow$ search for poles in the complex energy plane of $T$
Putting different analyses on common ground
... and have eventually convergence of results among different groups (?)

- BnGa, Jülich, MAID, Zagreb... fit to SAID elastic $\pi N$ partial waves or use it as FSI.
- Single-energy solutions of SAID: narrow structures & consistency, but little statistical meaning.
- No control of the statistical impact of elastic $\pi N$ data on multi-reaction fits, performed by other groups.
- Q: is it possible to provide a simple to use interface, such that other groups can fit to SAID elastic $\pi N$ PWs and get the same $\chi^2$ as if they had fitted to the elastic $\pi N$ data?
Bands: Monte-Carlo error propagation using bootstrap (energy-dependent fit).

Statistically meaningful

but: correlations between different partial waves/multipoles are also there!

How to provide? **Solution**: Instead of PWs, we plan to provide correlation matrices for the SES.
Work in progress: Correlated $\chi^2$ fits

Fit “observable” $O(E)$ with

$$O(E) = (\text{PW 1})E + (\text{PW 2}) \frac{E^2}{100}$$

Data generated around

$$O(E) = E + \frac{E^2}{100}$$

- Stochastic estimate of correlation matrix.
- Correlated $\chi^2$: $\chi^2(x) = (x - \bar{x})^T C^{-1} (x - \bar{x})$, $x = (S_{11}, S_{31}, \ldots)$
- but does this provide the same $\chi^2$ as a fit to the original data?
... almost; the function

$$\chi^2 = (x - \bar{x})^T C^{-1} (x - \bar{x}) + \chi^2_{\text{best}} + \text{corrections}$$

does.

- Difference between correlated $\chi^2$ and actual $\chi^2$ of the data themselves (toy model):
Data base and fit

Data base:
- elastic $\pi N$ PWA SAID 2006 [Arndt et al., PRC 74 (2006)]
- $\pi N \rightarrow \eta N, KY$ observables ($d\sigma/d\Omega, P, \beta$)

Fit parameters:
- $T^P$: 128 free parameters
  - 11 $N^*$ resonances $\times$ (1 $m_{\text{bare}}$ + couplings to $\pi N, \rho N, \eta N, \pi \Delta, K\Lambda, K\Sigma$) + 10 $\Delta$ resonances $\times$ (1 $m_{\text{bare}}$ + couplings to $\pi N, \rho N, \pi \Delta, K\Sigma$)
- $T^{NP}$: 68 free parameters

$\Rightarrow$ Simultaneous fit of $T^P$ and $T^{NP}$ using MINUIT on the JUROPA supercomputer (FZJ)

Sensitivity of results to starting conditions?

$\Rightarrow$ Perform 2 fits, start from different scenarios in parameter set

- Fit A: start from parameter set of a preceding Jülich model [Gasparyan et al. PRC68, (2003)]
- Fit B: start from a new scenario
**Photoproduction in a semi-phenomenological approach**

### Multipole amplitude

\[ M_{\mu \gamma}^{IJ} = V_{\mu \gamma}^{IJ} + \sum_{\kappa} T_{\mu \kappa}^{IJ} G_{\kappa} V_{\kappa \gamma}^{IJ} \]

\( (\text{partial wave basis}) \)

\[ T_{\mu \kappa}: \text{Jülich hadronic } T\text{-matrix} \rightarrow \text{Watson's theorem fulfilled by construction} \]

\[ \rightarrow \text{analyticity of } T: \text{extraction of resonance parameters} \]

### Phenomenological potential:

\[ V_{\mu \gamma}(E, q) = \frac{\tilde{\gamma}_{\mu; i}^a(q)}{m_N} P_{\mu}^{NP}(E) + \sum_i \frac{\gamma_{\mu; i}^a(q) P_i^P(E)}{E - m_i^P} \]

\( \tilde{\gamma}_{\mu; i}^a, \gamma_{\mu; i}^a: \text{hadronic vertices} \rightarrow \text{correct threshold behaviour} \)

\( i: \text{resonance number per multipole; } \mu: \text{channels } \pi N, \eta N, \pi \Delta \)

Method inspired by SAID but different
Multipoles: Comparison with GWU/SAID CM12

Re(A) [10^{-3} fm]

Im(A) [10^{-3} fm]

E [GeV]
Multipoles: Comparison with GWU/SAID CM12
Multipoles: $E_{0+} (\pi^0 p)$ at threshold

\[ M_{\pi^0 p}^\gamma = \gamma + \pi^0 p + \pi^0 p + \gamma + \pi^0 p + \pi^+ n + \pi^0 p + \ldots \]

Different $\pi$ & $N$ masses → cusp effect
Details of the formalism

Polynomials:

\[ P^P_i(E) = \sum_{j=1}^{n} g^{P}_{i,j} \left( \frac{E - E_0}{m_N} \right)^j e^{-g^{P}_{i,n+1}(E-E_0)} \]

\[ P^{NP}_\mu(E) = \sum_{j=0}^{n} g^{NP}_{\mu,j} \left( \frac{E - E_0}{m_N} \right)^j e^{-g^{NP}_{\mu,n+1}(E-E_0)} \]

- \( E_0 = 1077 \text{ MeV} \)
- \( g^{P}_{i,j}, g^{NP}_{\mu,j} \): fit parameter
- \( e^{-g(E-E_0)} \): appropriate high energy behavior
- \( n = 3 \)
Isospin breaking in the $\pi N$ channel

- Hadronic reactions: data well above threshold → isospin averaged masses
- Pion-photoproduction: data near threshold → include isospin breaking
  $\leftrightarrow$ use exact $\pi^0$ ($\pi^+$) and $p$ ($n$) mass

$E_{cm} < 1140$ MeV:

$$M_{\pi^0 p}^\gamma = \gamma \rightarrow p \pi^0 \rightarrow p p \pi^0 + \ldots$$

$$M_{\pi^+ n}^\gamma = \gamma \rightarrow p \pi^+ \rightarrow p n \pi^0 + \ldots$$
Isospin breaking in the $\pi N$ channel

- Hadronic reactions: data well above threshold $\rightarrow$ isospin averaged masses
- Pion-photoproduction: data near threshold $\rightarrow$ include isospin breaking
  $\Rightarrow$ use exact $\pi^0$ ($\pi^+$) and $p$ ($n$) mass

$E_{cm} < 1140$ MeV:

\[ M_{\gamma \pi^0 p}^\gamma = \gamma + \ldots \]

\[ M_{\gamma \pi^+ n}^\gamma = \gamma + \ldots \]
$\pi N \rightarrow \pi N$: Partial wave amplitudes $I=1/2$
$\pi N \rightarrow \pi N$: Partial wave amplitudes $I=3/2$
Full results $\eta N: d\sigma/d\Omega$
Full results $\eta N: d\sigma/d\Omega$ & Polarization
\( \pi^- p \to \eta N: \) Total cross section

![Graph showing the total cross section for the reaction \( \pi^- p \to \eta N \). The graph plots the cross section \( \sigma \) in mb vs. the energy \( z \) in MeV. The data points are accompanied by error bars. The graph also indicates the \( K\Lambda \) and \( K\Sigma \) thresholds. The partial wave content is indicated as 22/54.]
$\eta/N$: Partial wave content

![Graph showing partial wave content of $\eta/N$ with various partial waves labeled P$_{11}$, S$_{11}$, D$_{13}$, D$_{15}$, F$_{15}$, F$_{17}$, G$_{17}$, and G$_{19}$.]
Full results $\mathcal{K}\Lambda$: $d\sigma/d\Omega$
Full results $K\Lambda: d\sigma/d\Omega$

\begin{align*}
\text{z} &= 1999 \text{ MeV} \\
&\quad 2026 \text{ MeV} \\
&\quad 2059 \text{ MeV} \\
&\quad 2097 \text{ MeV} \\
&\quad 2104 \text{ MeV} \\
&\quad 2137 \text{ MeV} \\
&\quad 2159 \text{ MeV} \\
&\quad 2182 \text{ MeV} \\
\text{z} &= 2208 \text{ MeV} \\
&\quad 2244 \text{ MeV} \\
&\quad 2259 \text{ MeV} \\
&\quad 2305 \text{ MeV} \\
&\quad 2316 \text{ MeV} \\
&\quad 2405 \text{ MeV} \\
\end{align*}

$\text{cos } \Theta$
Full results $K\Lambda$: Polarization
Full results $K\Lambda$: Spinrotation parameter $\beta$

$$\beta = \arctan \left( \frac{2 \text{Im}(h^*_f g_f)}{|g_f|^2 - |h|^2} \right)$$
$\pi^- p \rightarrow K^0 \Lambda$: Total cross section

$K\Lambda$: Partial wave content
Full results $K^0\Sigma^0$: $d\sigma/d\Omega$

![Graphs showing the differential cross section $d\sigma/d\Omega$ for $K^0\Sigma^0$ interactions at various energies and cosines Θ. The graphs display data points with error bars and fitted curves for different energies, such as z = 1694 MeV, 1724 MeV, 1741 MeV, 1758 MeV, 1792 MeV, 1797 MeV, 1819 MeV, 1825 MeV, 1845.5 MeV, 1879 MeV, 1909 MeV, 1930 MeV, 1934 MeV, 1938 MeV, 1966 MeV, 1978 MeV, z = 1999 MeV, 2026 MeV, 2059 MeV, 2097 MeV, 2104 MeV, 2137 MeV, 2182 MeV, 2208 MeV, z = 2244 MeV, 2259 MeV, 2305 MeV, 2316 MeV, 2405 MeV, and 2405 MeV for different cosines Θ.]
Full results $K^0\Sigma^0$: Polarization

[Graph showing data points and curves for different energies, with axes labeled $P$, $\cos\theta$, and energy values like 1694 MeV, 1724.5 MeV, etc.]
$K^0\Sigma^0$: Partial wave content $I=1/2$
"$K^0 \Sigma^0$: Partial wave content $l=3/2$"
Full results $K^+\Sigma^-: d\sigma/d\Omega$

no Polarization data!
$\pi^- p \rightarrow K^+ \Sigma^-$: Total cross section
Full results $K^+\Sigma^+: d\sigma/d\Omega$
Full results $K^+\Sigma^+$: Polarization

$P_z = 1729$ MeV 1732 MeV 1757 MeV 1764 MeV
1783 MeV 1789 MeV 1790 MeV 1813 MeV

$P_z = 1822$ MeV 1845 MeV 1870 MeV 1891 MeV

$P_z = 2019$ MeV 2031 MeV 2059 MeV 2074 MeV
2106 MeV 2118 MeV 2147 MeV 2158 MeV

$P_z = 2188$ MeV 2202 MeV 2224 MeV 2243 MeV
2261 MeV 2282 MeV 2304 MeV 2318 MeV

$P_z$ vs. $\cos\theta$ for $K^+\Sigma^+$ final states.
$\pi^+ p \rightarrow K^+ \Sigma^+$: Total cross section
Resonance content: I=1/2

Pole search on the 2\textsuperscript{nd} sheet of the scattering matrix \( T_{\mu\nu} \)

Resonance parameter:

- "mass" = \( \text{Re}(E_0) \)
- "width" = \(-2\text{Im}(E_0)\)
- Residues \( \rightarrow \) branching ratios

\( E_0 \): pole position

Notation: \( N(\text{"name"})^J^\text{parity} \)

\( \times \): branch points

Numbers
## Resonance content: $I = 1/2$

| fit $\rightarrow$ | Re $z_0$ | -2Im $z_0$ | $|r_{\pi N}|$ | $\theta_{\pi N \rightarrow \pi N}$ | $\Gamma_{\pi N}/\Gamma_{\text{tot}}$ |
|-----------------|----------|------------|----------------|------------------|------------------|
|                 | [MeV]    | [MeV]      | [MeV]          | [deg]            | [%]              |
|                 | A        | B          | A              | B               | A               | B               |
| $N(1535)\ 1/2^-$ | 1498     | 1497       | 74             | 66              | 17              | 13              | -37             | -43             | 40              | 44              |
| $N(1650)\ 1/2^-$ | 1677     | 1675       | 146            | 131             | 45              | 27              | -43             | -38             | 61              | 41              |
| $N(1440)\ 1/2^+_{(a)}$ | 1353     | 1348       | 212            | 238             | 59              | 62              | -103            | -111            | 56              | 52              |
| $N(1710)\ 1/2^+$ | 1637     | 1653       | 97             | 112             | 4               | 8               | -30             | 34              | 8.2             | 14              |
| $N(1750)\ 1/2^+_{(*,a)}$ | 1742     | (nf)       | 318            | -               | 8               | -               | 161             | -               | 5.1             | -               |
| $N(1720)\ 3/2^+$ | 1717     | 1734       | 208            | 306             | 7               | 18              | -76             | -23             | 6.6             | 11.6            |
| $N(1520)\ 3/2^-$ | 1519     | 1525       | 110            | 104             | 42              | 36              | -16             | 10              | 79              | 69              |
| $N(1675)\ 5/2^-$ | 1650     | 1640       | 126            | 178             | 24              | 34              | -19             | -32             | 39              | 38              |
| $N(1680)\ 5/2^+$ | 1666     | 1667       | 108            | 120             | 36              | 41              | -24             | -24             | 67              | 68              |
| $N(1990)\ 7/2^+$ | 1788     | 1936       | 282            | 244             | 4               | 4               | -84             | -87             | 3.2             | 3.6             |
| $N(2190)\ 7/2^-$ | 2092     | 2054       | 363            | 486             | 42              | 44              | -31             | -57             | 23              | 18              |
| $N(2250)\ 9/2^-$ | 2141     | 2036       | 465            | 442             | 17              | 13              | -67             | -62             | 7.5             | 5.7             |
| $N(2220)\ 9/2^+$ | 2196     | 2156       | 662            | 565             | 87              | 46              | -67             | -72             | 26              | 16              |
Resonance content: $I = 1/2$ Branching ratios

<table>
<thead>
<tr>
<th></th>
<th>$\frac{\Gamma^{1/2}<em>{\pi N} \Gamma^{1/2}</em>{\eta N}}{\Gamma_{tot}}$</th>
<th>$\theta_{\pi N \rightarrow \eta N}$</th>
<th>$\frac{\Gamma^{1/2}<em>{\pi N} \Gamma^{1/2}</em>{K \Lambda}}{\Gamma_{tot}}$</th>
<th>$\theta_{\pi N \rightarrow K \Lambda}$</th>
<th>$\frac{\Gamma^{1/2}<em>{\pi N} \Gamma^{1/2}</em>{K \Sigma}}{\Gamma_{tot}}$</th>
<th>$\theta_{\pi N \rightarrow K \Sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N(1650) 1/2^-$</td>
<td>51 48 120 115</td>
<td>7.7 8.3 68 77</td>
<td>15 34 -74 -83</td>
<td>26 15 -63 -59</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1440) 1/2^+_{(a)}$</td>
<td>2 5 -40 -26</td>
<td>2 11 156 152</td>
<td>1 2 67 189</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1710) 1/2^+$</td>
<td>24 23 130 164</td>
<td>9.4 17 -83 -41</td>
<td>3.9 0.1 -136 -112</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1750) 1/2^+_{(*,a)}$</td>
<td>0.5 -140 - -</td>
<td>0.8 -170 - -</td>
<td>2.2 - 4 -</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1720) 3/2^+$</td>
<td>1.2 3.1 98 117</td>
<td>3.1 2.9 -89 -63</td>
<td>1.7 2.2 64 90</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1520) 3/2^-$</td>
<td>3.5 3.0 87 113</td>
<td>5.8 6.3 158 177</td>
<td>0.8 3.6 163 164</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1675) 5/2^-$</td>
<td>6.0 3.6 -40 -66</td>
<td>0.3 1.7 -93 -122</td>
<td>3.3 3.7 -168 -179</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1680) 5/2^+$</td>
<td>0.4 1.5 -47 -54</td>
<td>0.2 0.3 -99 72</td>
<td>0.1 0.1 141 141</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(1990) 7/2^+$</td>
<td>0.4 0.5 -99 90</td>
<td>1.7 1.5 -123 -99</td>
<td>0.8 1.2 28 70</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(2190) 7/2^-$</td>
<td>0.1 0.4 -28 99</td>
<td>1.9 0.3 -51 -75</td>
<td>1.3 0.5 -63 -105</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(2250) 9/2^-$</td>
<td>0.6 0.1 -92 -96</td>
<td>1.1 0.7 -103 -106</td>
<td>0.3 0.7 -114 62</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N(2220) 9/2^+$</td>
<td>0.1 0.3 63 74</td>
<td>0.9 0.8 53 59</td>
<td>0.8 0.1 -138 52</td>
<td>1.2 3.1 98 117</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Resonance content: \( I = 3/2 \)

| Resonance | State | \( \text{Re} \, z_0 \) [MeV] | \( -2 \text{Im} \, z_0 \) [MeV] | \( |r_{\pi N}| \) [MeV] | \( \theta_{\pi N \rightarrow \pi N} \) [deg] |
|---|---|---|---|---|---|
| \( \Delta(1620) \) | \( 1/2^- \) | 1599 | 1596 | 71 | 80 | 17 | 18 | -107 | -107 |
| \( \Delta(1910) \) | \( 1/2^+ \) | 1788 | 1848 | 575 | 376 | 56 | 20 | -140 | -143 |
| \( \Delta(1232) \) | \( 3/2^+ \) | 1220 | 1218 | 86 | 96 | 44 | 50 | -35 | -38 |
| \( \Delta(1600) \) | \( 3/2^+_{(a)} \) | 1553 | 1623 | 352 | 284 | 20 | 27 | -158 | -124 |
| \( [\Delta(1920) \) | \( 3/2^+ \) | 1724 | 1808 | 863 | 887 | 36 | 19 | 163 | -70 |
| \( \Delta(1700) \) | \( 3/2^- \) | 1675 | 1705 | 303 | 185 | 24 | 14 | -9 | -4 |
| \( \Delta(1930) \) | \( 5/2^- \) | 1775 | 1805 | 646 | 580 | 18 | 14 | -159 | 3 |
| \( \Delta(1905) \) | \( 5/2^+ \) | 1770 | 1776 | 259 | 143 | 17 | 9 | -59 | -40 |
| \( \Delta(1950) \) | \( 7/2^+ \) | 1884 | 1890 | 234 | 232 | 58 | 58 | -25 | -19 |
| \( \Delta(2200) \) | \( 7/2^- \) | 2147 | 2111 | 477 | 353 | 17 | 20 | -52 | 7 |
| \( \Delta(2400) \) | \( 9/2^- \) | 1969 | 1938 | 577 | 559 | 25 | 19 | -80 | -112 |
Resonance content: $I = 3/2$ Branching ratios

<table>
<thead>
<tr>
<th>Resonance</th>
<th>$\Gamma_{\pi N} / \Gamma_{\text{tot}}$</th>
<th>$\Gamma_{K \Sigma}^{1/2} / \Gamma_{\text{tot}}$</th>
<th>$\theta_{\pi N \rightarrow K \Sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta(1620) 1/2^-$</td>
<td>48 45</td>
<td>22 24</td>
<td>-107 -107</td>
</tr>
<tr>
<td>$\Delta(1910) 1/2^+$</td>
<td>20 10</td>
<td>4.7 1.9</td>
<td>-144 -115</td>
</tr>
<tr>
<td>$\Delta(1232) 3/2^+$</td>
<td>100 100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\Delta(1600) 3/2^+_{(a)}$</td>
<td>11 19</td>
<td>11 13</td>
<td>-7 41</td>
</tr>
<tr>
<td>$[\Delta(1920) 3/2^+]$</td>
<td>8.3 4.3</td>
<td>16 14</td>
<td>-20 50</td>
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<tr>
<td>$\Delta(1700) 3/2^-$</td>
<td>16 16</td>
<td>1.5 1.6</td>
<td>-150 -121</td>
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<tr>
<td>$\Delta(1930) 5/2^-$</td>
<td>5.6 4.8</td>
<td>3.1 1.7</td>
<td>-3 135</td>
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<tr>
<td>$\Delta(1905) 5/2^+$</td>
<td>13 12</td>
<td>0.5 0.1</td>
<td>-142 -99</td>
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<tr>
<td>$\Delta(1950) 7/2^+$</td>
<td>50 50</td>
<td>4.0 3.8</td>
<td>-78 -71</td>
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<tr>
<td>$\Delta(2200) 7/2^-$</td>
<td>7.2 11</td>
<td>0.6 0.1</td>
<td>-98 -33</td>
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<tr>
<td>$\Delta(2400) 9/2^-$</td>
<td>8.7 5.7</td>
<td>1.3 0.8</td>
<td>40 6</td>
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</table>
Data base

simultaneous fit to $\pi N \to \pi N, \eta N, K\Lambda, K\Sigma$

World data base on $\eta N, K\Lambda, K\Sigma$

<table>
<thead>
<tr>
<th>PWA</th>
<th>$\sigma_{tot}$</th>
<th>$\frac{d\sigma}{d\Omega}$</th>
<th>$P$</th>
<th>$\beta$</th>
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<tbody>
<tr>
<td>$\pi N \to \pi N$</td>
<td>GWU/SAID 2006</td>
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</tr>
<tr>
<td>$\pi^- p \to \eta n$</td>
<td>62 data points</td>
<td>38 energy points</td>
<td>12 energy points</td>
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</tr>
<tr>
<td></td>
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<tr>
<td>$\pi^- p \to K^0 \Lambda$</td>
<td>66 data points</td>
<td>46 energy points</td>
<td>27 energy points</td>
<td>7 energy points</td>
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<td></td>
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</tr>
<tr>
<td>$\pi^- p \to K^0 \Sigma^0$</td>
<td>16 data points</td>
<td>29 energy points</td>
<td>19 energy points</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\pi^- p \to K^+ \Sigma^-$</td>
<td>14 data points</td>
<td>15 energy points</td>
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<tr>
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<td></td>
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</tr>
<tr>
<td>$\pi^+ p \to K^+ \Sigma^+$</td>
<td>18 data points</td>
<td>32 energy points</td>
<td>32 energy points</td>
<td>2 energy pionts</td>
</tr>
</tbody>
</table>

$z = 1489$ to 2235 MeV
$1740$ to 2235 MeV

$1626$ to 1405 MeV
$1633$ to 2208 MeV
$1852$ to 2262 MeV

$1694$ to 2405 MeV
$1694$ to 2316 MeV

$1739$ to 2405 MeV

$1729$ to 2318 MeV
$1729$ to 2318 MeV
$2021$ and 2107 MeV

$\sim 6000$ data points
\( t \) and \( u \), contd.; \( s \)-channel exchanges
Potentials for resonances with $J > 3/2$

- Correct dependence on the orbital angular momentum (centrifugal barrier)

$$
\begin{align*}
\nu_{\gamma \frac{5}{2}^-} &= \frac{k}{M} \nu_{\gamma \frac{3}{2}^+} \\
\nu_{\gamma \frac{5}{2}^+} &= \frac{k}{M} \nu_{\gamma \frac{3}{2}^-} \\
\nu_{\gamma \frac{7}{2}^-} &= \frac{k^2}{M^2} \nu_{\gamma \frac{5}{2}^-} \\
\nu_{\gamma \frac{7}{2}^+} &= \frac{k^2}{M^2} \nu_{\gamma \frac{5}{2}^+}
\end{align*}
$$

$\gamma = \pi N, \rho N, \eta N, \pi \Delta, \sigma N, K \Lambda$ and $K \Sigma$

$M$ : mass of the baryon in the particular channel
Phenomenological potential:

\[
V_{\mu\gamma}(E, q) = \gamma^N \rightarrow \gamma^m + \gamma^N \rightarrow N^*, \Delta^* \rightarrow \gamma^m = \frac{\tilde{\gamma}_{\mu;i}^a(q)}{P^P_{\mu} \mu} \frac{P^{NP}_{\mu} (E)}{m_N} + \sum_i \frac{\gamma_{\mu;i}^a(q) P^P_i (E)}{E - m_i^b}
\]

\(\tilde{\gamma}_{\mu;i}^a, \gamma_{\mu;i}^a\): hadronic vertices \(\rightarrow\) correct threshold behaviour

\(i\): resonance number per multipole

\(\mu\): channels \(\pi N, \eta N, \pi \Delta\)

\(P^P_i, P^{NP}_{\mu}\): energy-dependent polynomials
Helicity multipoles $A^h_{L\pm}$:

- $J = L + 1/2$:
  
  $A^{1/2}_{L+} = -\frac{1}{2} [(L + 2) E_{L+} + L M_{L+}]$
  
  $A^{3/2}_{L+} = \frac{1}{2} \sqrt{L(L+2)} [E_{L+} - M_{L+}]$

- $J = L - 1/2$:

  $A^{1/2}_{L-} = -\frac{1}{2} [(L - 1) E_{L-} - (L + 1) M_{L-}]$

  $A^{3/2}_{L-} = -\frac{1}{2} \sqrt{(L - 1)(L + 1)} [E_{L-} + M_{L-}]$
Matching to lattice
Prediction & analysis of lattice data

[Matching to lattice
Prediction & analysis of lattice data

[M. Döring et al., EPJ A47, 163 (2011)]

Scattering equation:

\[ T(q'', q') = V(q'', q') + \int_{0}^{\infty} dq \frac{1}{q^2} V(q'', q) \frac{1}{z - E_1(q) - E_2(q) + i\epsilon} T(q, q') \]

Discretization of momenta in the scattering equation:

\[ \int \frac{\bar{d}^3 q}{(2\pi)^3} f(|\bar{q}|^2) \rightarrow \frac{1}{L^3} \sum_{\bar{n}_i} f(|\bar{q}_i|^2), \quad \bar{q}_i = \frac{2\pi}{L} \bar{n}_i, \quad \bar{n}_i \in \mathbb{Z}^3 \]

\[ T(q'', q') = V(q'', q') + \frac{2\pi^2}{L^3} \sum_{i=0}^{\infty} \vartheta(i) V(q'', q_i) \frac{1}{z - E_1(q_i) - E_2(q_i)} T(q_i, q'), \]

\( \vartheta^{(P)}(i) \) series

- Study finite size effects
- Predict lattice spectra
Two-body scattering
in the infinite volume limit

- Unitarity of the scattering matrix $S$: $SS^\dagger = 1$

$$ [S = 1 - i \frac{p}{4\pi E} T]. $$

$$ \text{Im } T^{-1}(E) = \sigma \equiv \frac{p}{8\pi E} $$

- $\rightarrow$ Generic (Lippman-Schwinger) equation for unitarizing the $T$-matrix:

$$ T = V + V G T \quad \text{Im } G = -\sigma $$

$V$: (Pseudo)potential, $\sigma$: phase space.

- $G$: Green’s function:

$$ G = \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2 + i\epsilon}, $$

$$ \omega_{1,2}^2 = m_{1,2}^2 + \vec{q}^2 $$
Discretization
Discretized momenta in the finite volume with periodic boundary conditions

\[ \Psi(\vec{x}) \xleftarrow{\hat{L}} \Psi(\vec{x} + \hat{e}_i L) = \exp (i L q_i) \Psi(\vec{x}) \implies q_i = \frac{2\pi}{L} n_i, \quad n_i \in \mathbb{Z}, \quad i = 1, 2, 3 \]

\[
\int \frac{d^3 \vec{q}}{(2\pi)^3} g(|\vec{q}|^2) \rightarrow \frac{1}{L^3} \sum_{\vec{\eta}} g(|\vec{q}|^2), \quad \vec{q} = \frac{2\pi}{L} \vec{\eta}, \quad \vec{\eta} \in \mathbb{Z}^3
\]

\[ G \rightarrow \tilde{G} = \frac{1}{L^3} \sum_{\vec{q}} \frac{f(|\vec{q}|)}{E^2 - (\omega_1 + \omega_2)^2} \]

\[ \bullet \quad E > m_1 + m_2: \ \tilde{G} \text{ has poles at free energies in the box, } E = \omega_1 + \omega_2 \]

\[ \bullet \quad E < m_1 + m_2: \ \tilde{G} \rightarrow G \text{ exponentially with } L \text{ (regular summation theorem).} \]
Finite $\rightarrow$ infinite volume: the Lüscher equation

Warning: Very crude re-derivation.

- Measured eigenvalues of the Hamiltonian (tower of lattice levels $E(L)$) → Poles of scattering equation $\tilde{T}$ in the finite volume $\rightarrow$ determines $V$:

  $$\tilde{T} = (1 - V\tilde{G})^{-1}V \rightarrow V^{-1} - \tilde{G} \doteq 0 \rightarrow V^{-1} = \tilde{G}$$

- The interaction $V$ determines the $T$-matrix in the infinite volume limit:

  $$T = \left(V^{-1} - G\right)^{-1} = \left(\tilde{G} - G\right)^{-1}$$

- Re-derivation of Lüscher’s equation ($T$ determines the phase shift $\delta$):

  $$p \cot \delta(p) = -8\pi\sqrt{s} \left(\tilde{G}(E) - \text{Re} G(E)\right)$$

- $V$ and dependence on renormalization have disappeared (!)
- $p$: c.m. momentum
- $E$: scattering energy
- $\tilde{G} - \text{Re} G$: known kinematical function
  ($\simeq Z_{00}$ up to exponentially suppressed contributions)
- One phase at one energy.
Connecting phenomenology to Lattice QCD

- **Side length** $L$, periodic boundary conditions
  \[ \Psi(x) = \Psi(x + \hat{e}_i L) \]
  \[ \rightarrow \text{finite volume effects} \]
  \[ \rightarrow \text{Infinite volume } L \rightarrow \infty \text{ extrapolation} \]

- **Lattice spacing** $a$
  \[ \rightarrow \text{finite size effects} \]
  Modern lattice calculations:
  \[ a \approx 0.07 \text{ fm} \rightarrow p \approx 2.8 \text{ GeV} \]
  \[ \rightarrow \text{(much) larger than typical hadronic scales;} \]
  not considered here.

- **Unphysically large quark/hadron masses**
  \[ \rightarrow \text{(chiral) extrapolation required} \]