Saint-Petersburg State University
Faculty of Applied Mathematics and Control Processes

Testing of symplectic integrator of spin-orbit motion based on matrix formalism

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Outline

- Brief introduction in matrix map integration.
- Mathematical model validation.
- Comparison of calculation with COSY Infinity.
- Fringe fields approach.
- Magnetic optics comparison.

Differential Algebra and Matrix Formalism

$$\frac{d}{dt}X = F(t, X); \quad X = M \circ X_0$$

for map building different approaches can be used:

$$X = \sum_{k=0}^{k} \frac{f^k(t_0, X_0)}{k!}(t - t_0)^k; \quad X = \int_{t_0}^{t} F(t, X)dt; \quad \frac{dM}{dt} = \tilde{F}(t, M)$$

COSY Infinity:

- differential algebra technique;
- reference orbit realignment;

MODE:

- matrix integration approach;
- zero-order part in a map;

Spin-orbit motion

State of a particle:

$$
\begin{aligned}
x_1 &= x, & x_2 &= y, x_3 = t,\\
x_4 &= \frac{p_x}{p_0}, & x_5 &= \frac{p_y}{p_0}, x_6 = \delta T,\\
x_7 &= S_x, & x_8 &= S_y, x_9 = S_s,
\end{aligned}
$$

Newton-Lorenz and T-BMT equations

$$
\begin{aligned}
p'_\xi =& p_\xi \left(\frac{v'}{v} + \frac{\gamma'}{\gamma}\right) + m_0 v\gamma \frac{\xi''}{H} - p_\xi \left(\frac{p_x}{m_0 v\gamma}\frac{x''}{H} + \frac{p_y}{m_0 v\gamma}\frac{y''}{H} + \frac{h_s h'_s}{H^2}\right),\\
S'_x =& \frac{\partial h_s}{\partial x} S_s + \frac{H}{v}\left[\Omega \times S\right]_x, S'_y = \frac{H}{v}\left[\Omega \times S\right]_y, S'_s = -\frac{\partial h_s}{\partial x} S_x + \frac{H}{v}\left[\Omega \times S\right]_s,\\
\Omega =& -\frac{q}{m_0\gamma}\left((1+\gamma G)B - \frac{G}{1+\gamma}\frac{p(p\cdot B)}{m_0^2 c^2} - (G + \frac{1}{1+\gamma})\frac{p\times E}{m_0 c^2}\right),
\end{aligned}
$$

Model verification

1) $B = (0; B_0; 0)$:

$$S'_x = -\frac{\gamma G}{R} S_z,$$
$$S'_z = +\frac{\gamma G}{R} S_x.$$

$$w = \gamma G$$

2) $B = (0; 0; B_z)$:

$$S'_x = +\frac{B_z}{B\rho}(1+G),$$
$$S'_y = -\frac{B_z}{B\rho}(1+G).$$

$$w = \frac{B_z}{B_0}(1+G)$$

2) $E = (E_0; 0; 0)$:

$$S'_x = -\frac{\gamma\beta^2}{R}\left(G - \frac{1}{\gamma^2 - 1}\right),$$
$$S'_z = +\frac{\gamma\beta^2}{R}\left(G - \frac{1}{\gamma^2 - 1}\right).$$

$$G = \frac{1}{\gamma_{mag}^2 - 1}$$

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Period of spin oscillation, sec

Case	COSY Infinity	MODE
1. Cylindrical deflector		
$\delta T = 1 \cdot 10^{-4}$	5749,4	5749,0
$\delta T = 3 \cdot 10^{-4}$	635,6	635,5
$\Delta x = 0,003$	1184,3	1184,3
2. Cylindrical deflector $\times$ 16		
$\delta T = 1 \cdot 10^{-4}$	5705,1	5704,6
$\delta T = 3 \cdot 10^{-4}$	633,9	633.8
3. Lattice with deflectors and quadrupole lenses		
$\delta T = 1 \cdot 10^{-4}$	0,2008	0,2008
$\delta T = 3 \cdot 10^{-4}$	0,0704	0,0704
$\Delta x = 0,003$	2072,3	2072,3
4. Lattice with deflectors, quadrupole lenses and RF		
$\delta T = 1 \cdot 10^{-4}$	4438,2	4415,3
$\delta T = 3 \cdot 10^{-4}$	492,9	491,7

Symplectic error estimation

Definition of a symplectic map

A map is symplectic if $M^* JM = J, \forall X_0,$, where $M = \partial X / \partial X_0$ and M^* is the transpose of M, E is identity matrix, $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$.

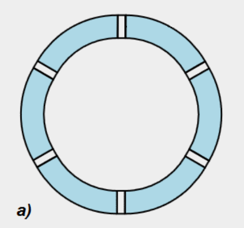
The error is calculated as a norm of matrix $\|M^* JM - J\|$.

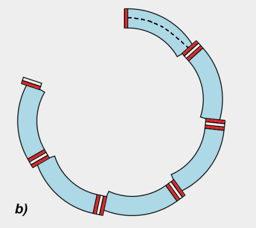
$$\frac{d}{dt} M = \tilde{F}(t, M) \Rightarrow$$

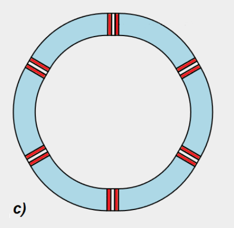
Method\step	$h = 0.2L$	$h = 0.1L$	$h = 0.01L$
Euler method	0.2233	0.1065	0.0104
Runge–Kutta 4th	0.0717	0.0205	0.0119
Implicit 2 stage Runge–Kutta 4th	0.0021	0.0004	0.0004

L means length of an element, h is an integration step.

Fringe field modelling

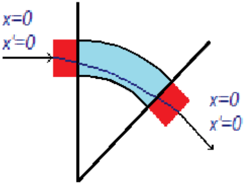


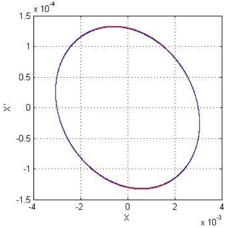


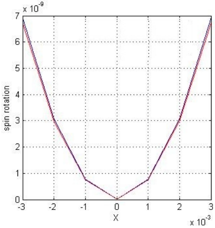


a) Fringe fields affects on reference orbit that is displaced in space.
b) COSY Infinity realigns the sequence of elements in order to connect inputs and outputs of adjacent elements.
c) MODE does not change the design orbit of the ring. You have to change field strength to obtain a reference orbit.

Fringe field modelling







$$X = R^0 + \sum_{i=1}^{k} R^i X_0^{[i]}$$

- Cylindrical deflector without fringe fields:

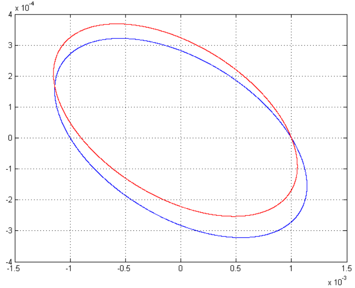
$$E_0 = 170 kV/cm,\ T = 232.79 MeV,\ R^0 \equiv 0$$

- Cylindrical deflector with fringe fields:

$$E_0 = 169 kV/cm,\ T = 232.90 MeV,\ R^0 \approx 0$$

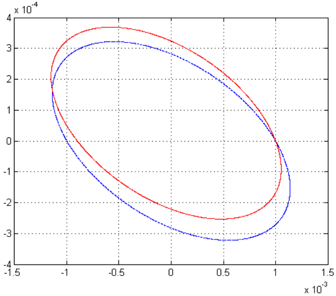
A sample of FODO structure and an element's axis offset

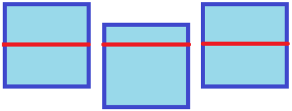
COSY Infinity:



SA -0.001 0 SA 0.001 0

MODE:

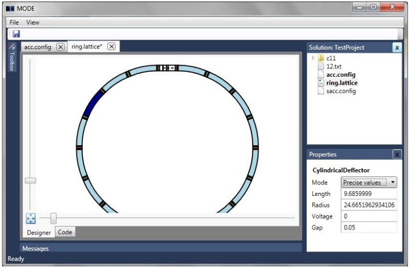


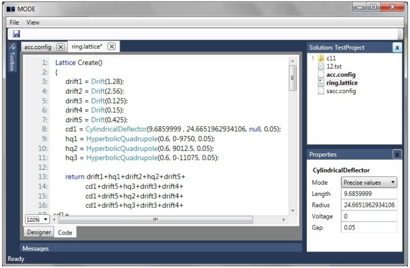


$\tilde{B} = B - \{0.001; 0; 0\}$

MODE: an IDE for simulation

Numerical method based of the matrix approach have been implemented. But there is only closed Beta version of the program:
a beta software release is a version of an application which is incomplete, and is supposed to perform as would the final version, but without any guarantee of sustained or intended functionality.





it have to be checked and verified carefully.

Thank you for your attention