On quantitative predictions for gravitational systematics in frozen spin storage rings

András László
laszlo.andras@wigner.hu
Wigner Research Centre for Physics, Budapest

*Class.Quant.Grav.***35**(2018)175003 and **arXiv2009.09820**

744th WE-Heraeus-Seminar, 31th March 2021

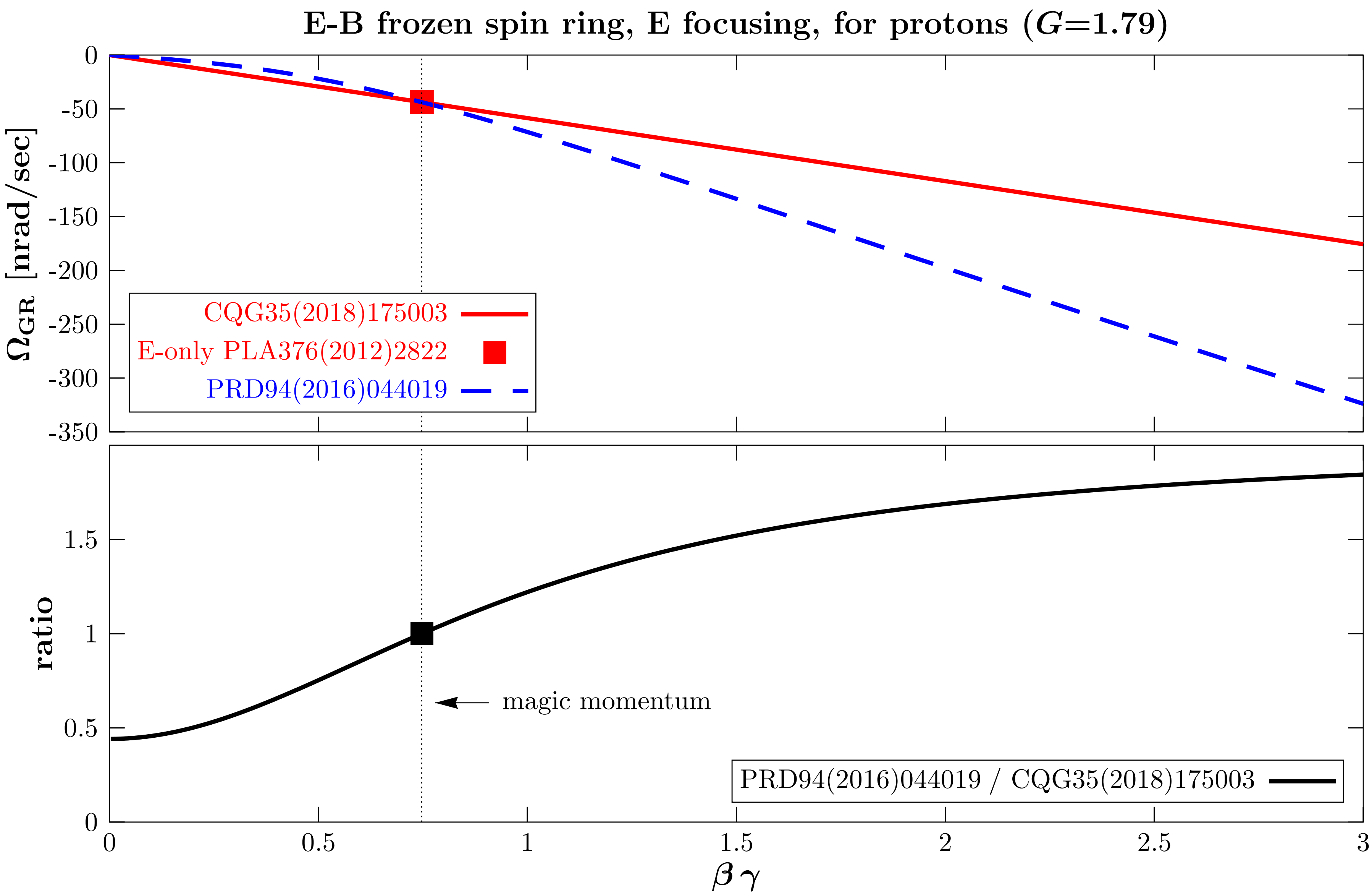
Introduction

- EDM observable in frozen spin storage ring: polarization “roll rate”.
- Many environmental factors give systematics to this observable.
- Most important one: the field imperfections, mostly of magnetic field.
- At $10^{-29}\, e{\cdot}\mathrm{cm} \;\; (\sim 10^{-9}\,\mathrm{rad/sec})$ sensitivity, gravitational signal will be also contributing. (see also talk of N.Nikolaev from Monday)
- There are slightly different quantitative predictions for the GR signal (differing by $\times$2–8). The aim of this talk is to clarify these.

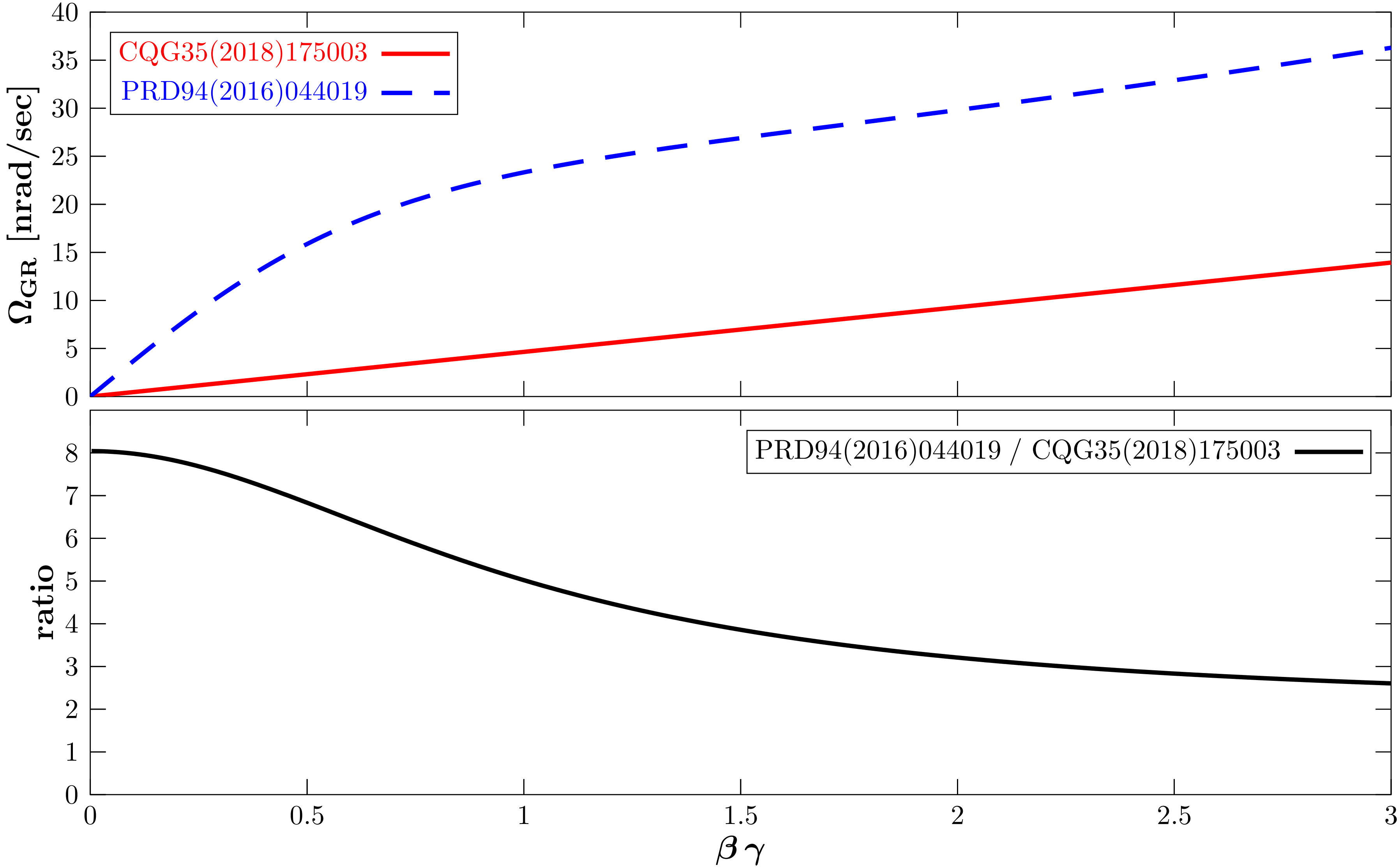
Predictions

$\approx 33\,\mathrm{nrad/sec}$

	method	Ω_{GR} ***prediction (*** E ***focusing)***
Silenko, Terayev: *Phys.Rev.* **D76**(2007)061101	perturbative lab frame formalism	first warning about possible GR effect in EDM ring, $\Omega_{\mathrm{GR}} \sim \frac{g}{c}$
Orlov, Flanagan, Semertzidis: *Phys.Lett.* **A376**(2012)2822	manifestly covariant weak field approximation	$\Omega_{\mathrm{GR}}\Big\vert_{\text{magic mom. (E-only ring)}} = -\sqrt{G}\,\frac{g}{c}$
Obukov, Silenko, Terayev: *Phys.Rev.* **D94**(2016)044019	perturbative lab frame formalism	$\Omega_{\mathrm{GR}} = \frac{(1-G\,(2\,\gamma^2-1))\,\beta}{\gamma}\,\frac{g}{c}$
László, Zimborás: *Class.Quant.Grav.* **35**(2018)175003	manifestly covariant nonperturbative full GR	$\Omega_{\mathrm{GR}} = -G\beta\gamma\,\frac{g}{c}$

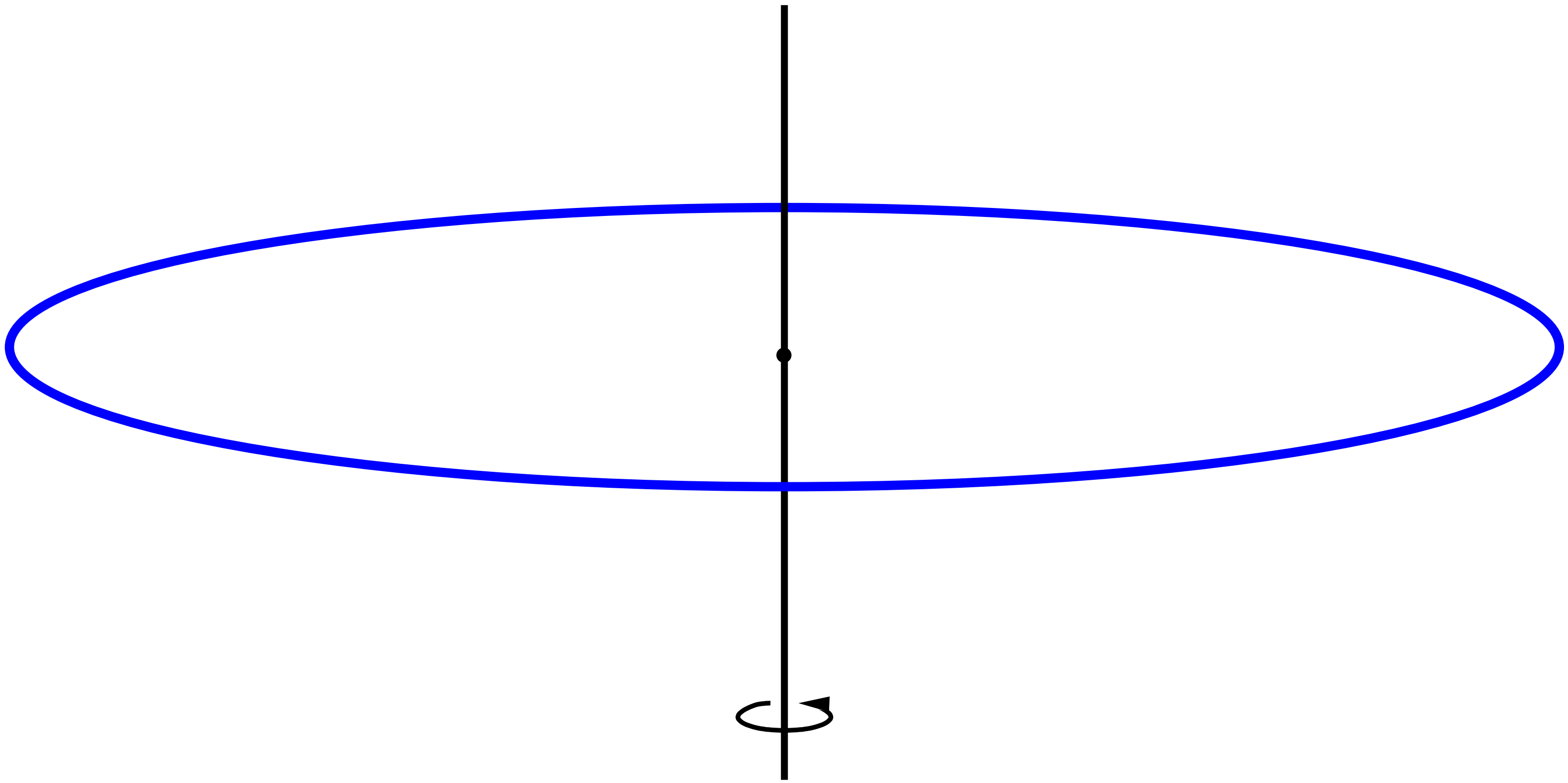

E-B frozen spin ring, E focusing, for protons (G=1.79)
Ω_GR [nrad/sec]
0
-50
-100
-150
-200
-250
-300
-350
CQG35(2018)175003
E-only PLA376(2012)2822
PRD94(2016)044019
ratio
1.5
1
0.5
0
magic momentum
PRD94(2016)044019 / CQG35(2018)175003
0
0.5
1
1.5
2
2.5
3
β γ

E-B frozen spin ring, E focusing, for deuterons (G=-0.142)



General remark:

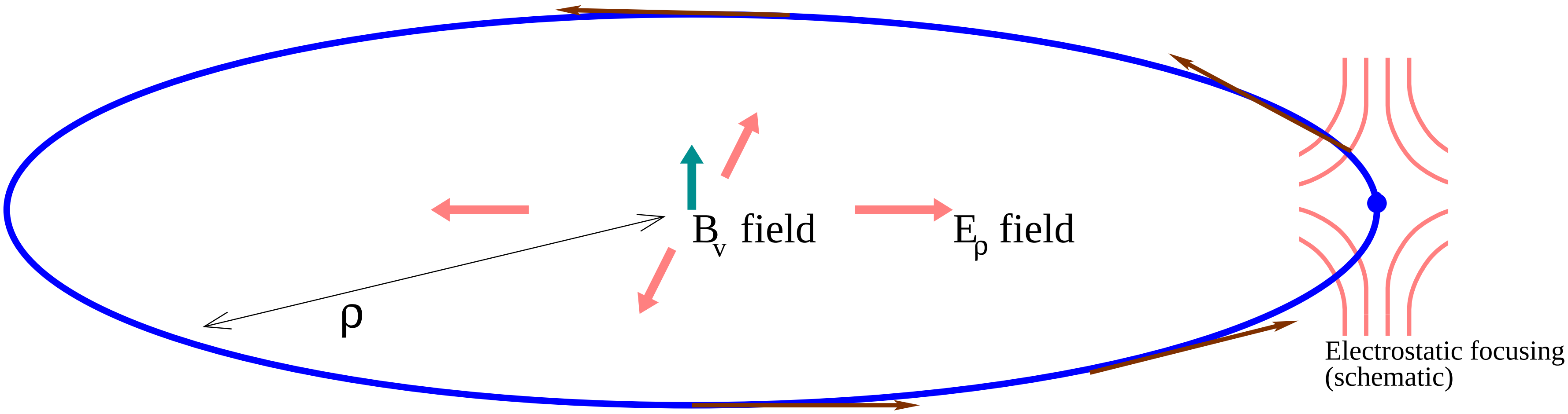
- Will speak only about effects for stationary planar circular idealized beam, for brevity. (Only basic phenomenon, not detailed imperfection systematics study, like C.Carli *et al.*)



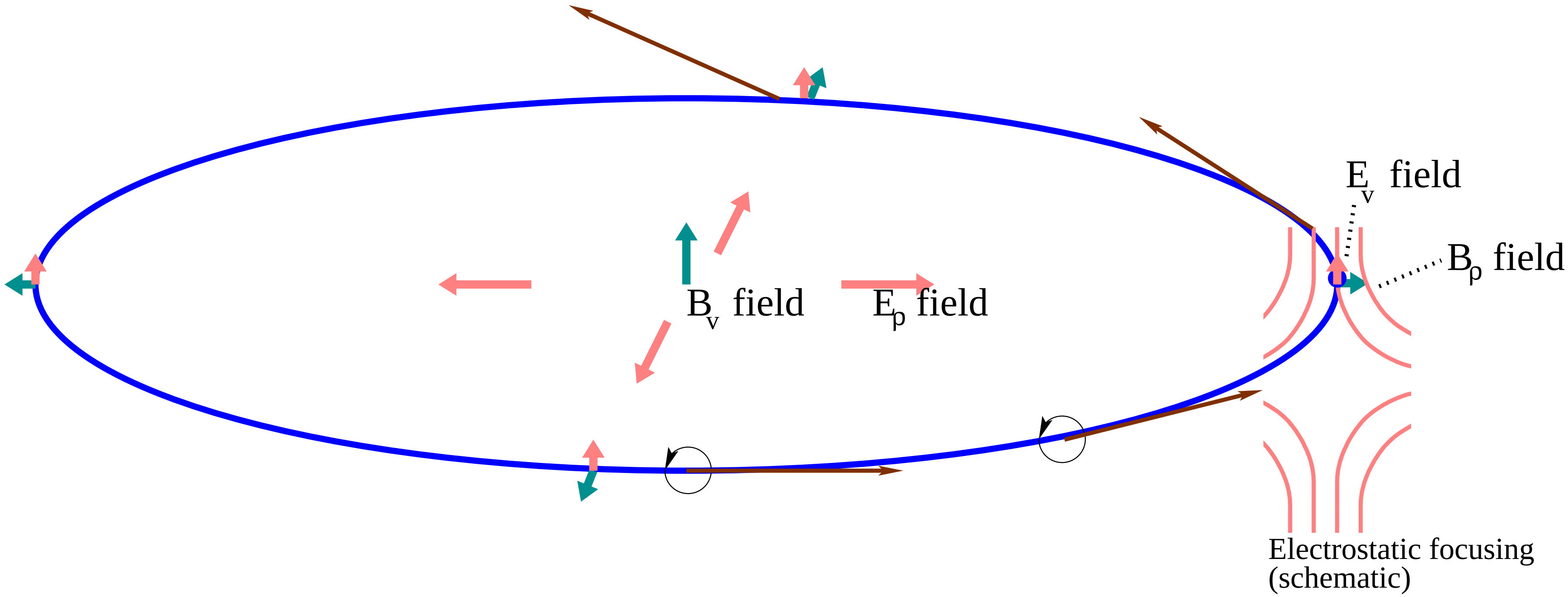
- Anyway this is necessary to clarify before one can consider generic beam systematics.

Notational conventions

Frozen spin ring:



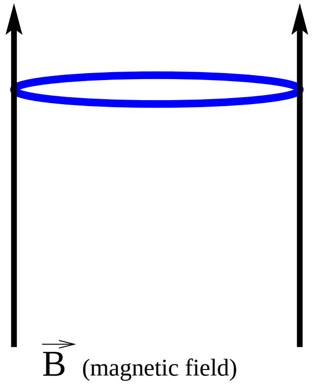
When B_ρ also present;



– planar circular beam orbit is satisfied,

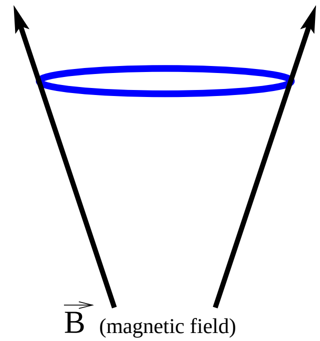
– spin precession due to magnetic moment is stopped *only in the bending plane* (!)

Spin “rolls” around instantaneous beam-radial axis, Koop spin wheel.



(perfect)

$$\xrightarrow{B_\rho}$$



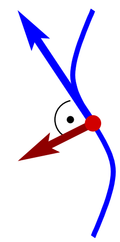
(imperfect)

$$\delta\Omega_{\text{roll}} \quad = \quad \underbrace{-\frac{q\,(1+G)}{m}\,\frac{1}{\gamma^2}\,B_\rho}_{\text{magnetic field shape imperfection term}}$$

(see also I.Koop: *Proc.IPAC2013*(2013)TUPWO040)

GR considerations

Relativistic motion of point particle with spin in electromagnetic and grav field:



u^a four velocity of at points of particle worldline.

s^a spin direction four vector at points of particle worldline.

(For quantum mechanical reasons: $g_{ab}\, u^a\, s^b = 0$.)

Equation of motion is Newton + Thomas-Bargmann-Michel-Telegdi equation.

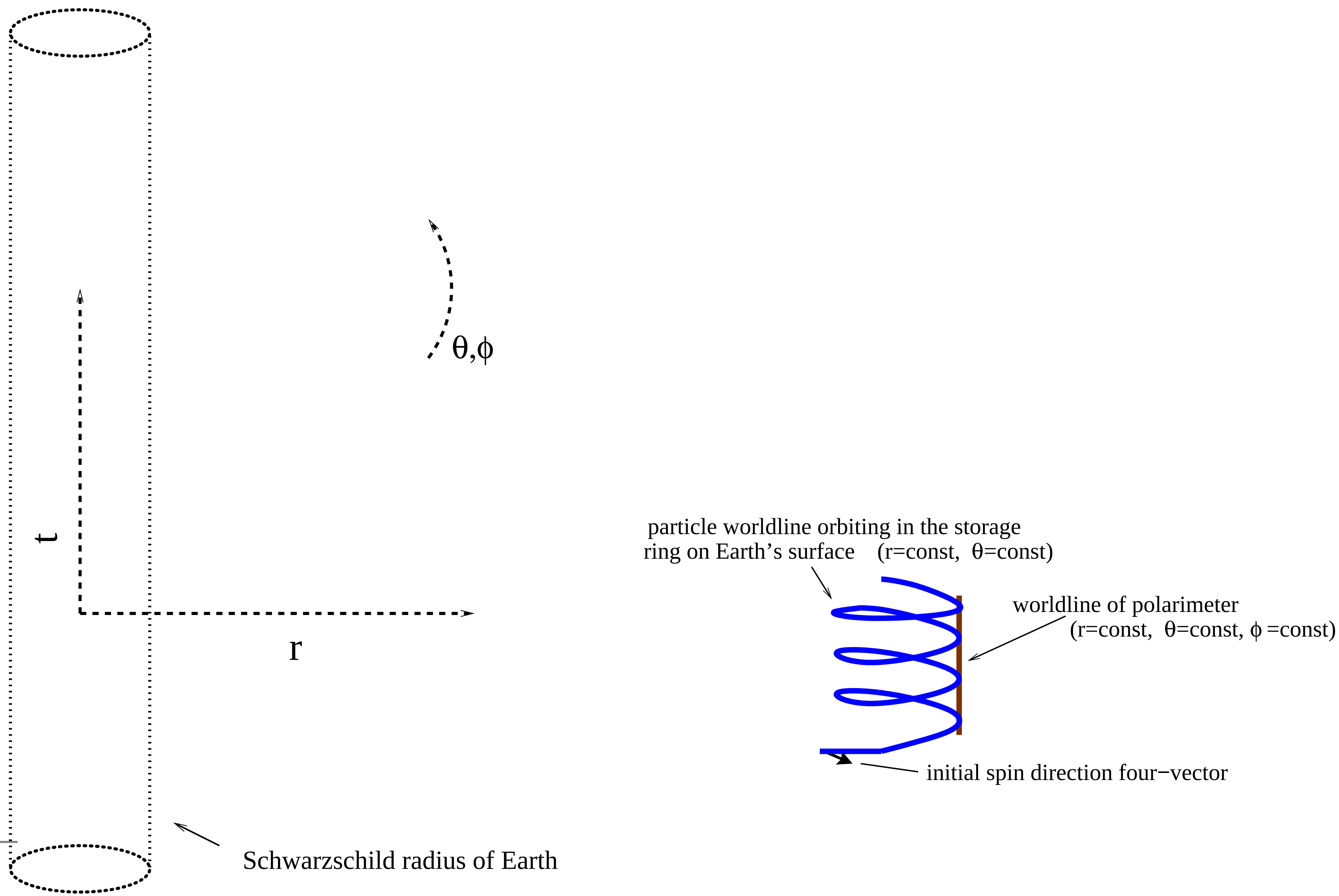
$$u^a \nabla_a u^b = -\frac{q}{m}\, F^{bc}\, u_c \qquad (\leftarrow \text{ Newton equation with EM force}),$$

$$D^F_u\, s^b = -\frac{\mu}{S}\left(F^{bc} - u^b u_d\; F^{dc} - \; F^{bd}\, u_d u^c \right) s_c \qquad (\leftarrow \text{ TBMT equation})$$
$$+\frac{d}{S}\left({}^\star F^{bc} - u^b u_d\, {}^\star F^{dc} - {}^\star F^{bd}\, u_d u^c \right) s_c . \qquad (\leftarrow\!-\!- \textit{ EDM term})$$

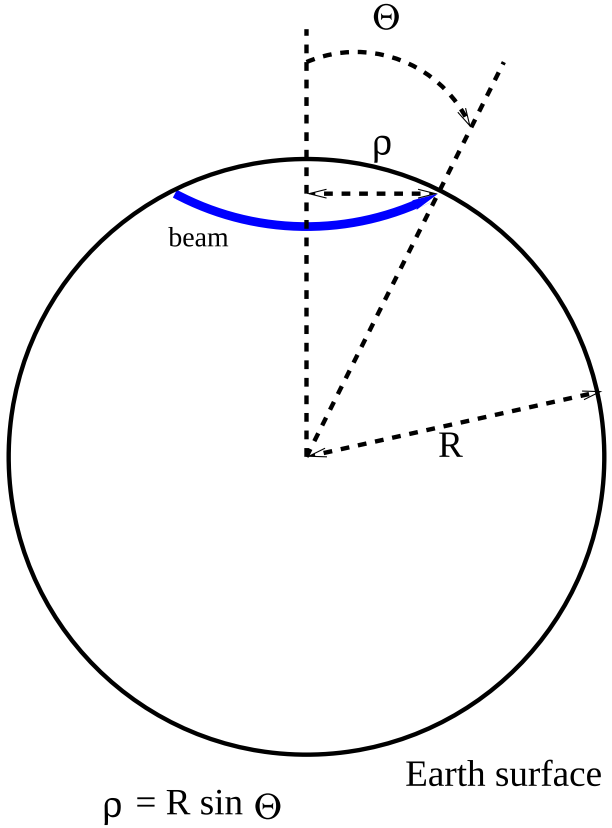
$D^F_u\, s^b = u^a \nabla_a s^b + (u^b\, u^a \nabla_a u_c)\, s^c - (u_c\, u^a \nabla_a u^b)\, s^c$ is the Fermi-Walker derivative.
(Conserves $g_{ab}\, u^a\, s^b = 0$. Free gyroscope equation is $D^F_u\, s^b = 0$.)

The kinematic configuration in GR setting

[In GR, there is no globally valid model for homogeneous grav. field → use Schwarzschild.]



Coordinate conventions: Schwarzschild metric:



$$g_{\mathsf{ab}}(t, r, \vartheta, \varphi) =$$

$$\begin{pmatrix} 1 - \frac{r_S}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_S}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\,\sin^2\vartheta \end{pmatrix}$$

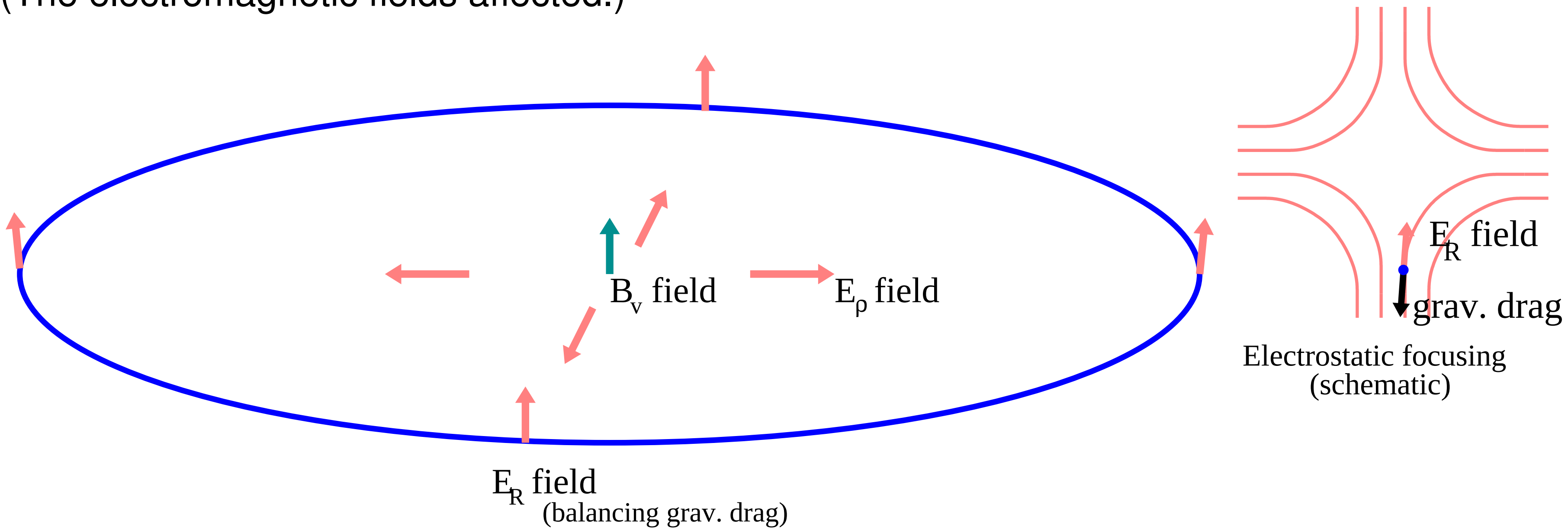
$$\left(r_S = \frac{2\,G_{\text{Newton}}\,M_{\text{Earth}}}{c^2} \approx 9\,\text{mm} \right)$$

Earth surface at: $r = R = const$.

Storage ring at: $r = R = const, \;\vartheta = \Theta = const$.

What does the GR modify?

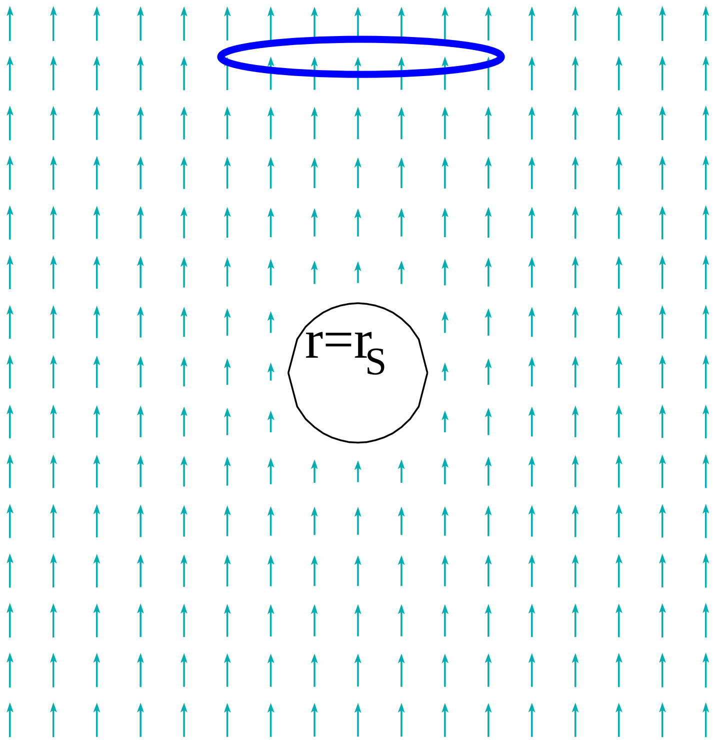
- The spacetime metric, and thus the parallel transport ∇_a, Fermi-Walker derivative D^F. (Newton and TBMT equations affected.)
- The Maxwell equations $\nabla_a F^{ab} = 0, \ \nabla_a {}^\star F^{ab} = 0$. (The electromagnetic fields affected.)



– "vertical homogeneous magnetic field",
– "beam-radial (outward cylindrical) electric field",
– "Earth-radial electric field"

make sense and calculable over Schwarzschild.

Magnetic bending field:

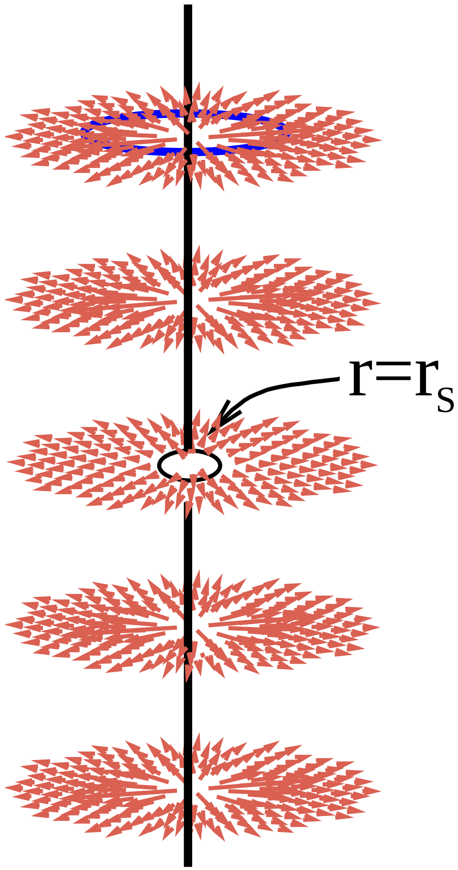


$$B_{\mathrm{v}}{}^{a}(t,r,\vartheta,\varphi) \quad \sim \quad \sqrt{1-\frac{r_S}{r}}\begin{pmatrix} 0 \\ \cos\vartheta \\ -\frac{1}{r}\,\sin\vartheta \\ 0 \end{pmatrix}$$

Field inside infinitely big solenoid.

(Asymptotically homogeneous magnetic field, superimposed on a massive body.)

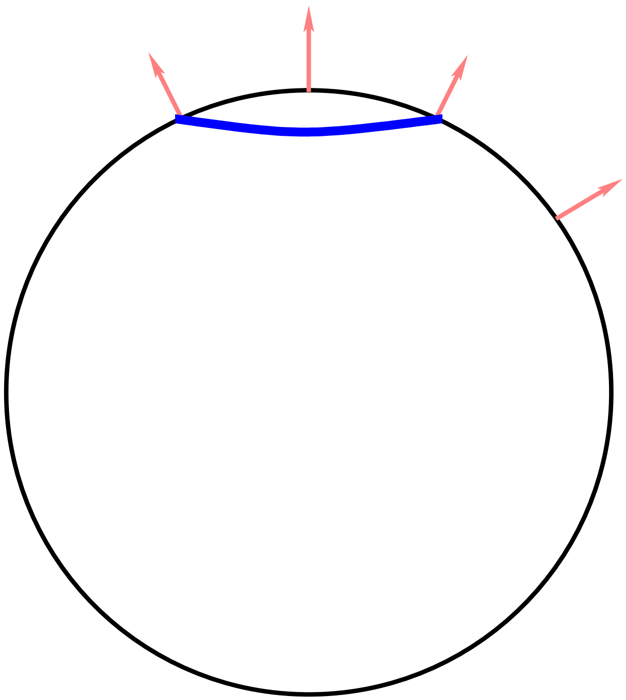
Electrostatic bending field:

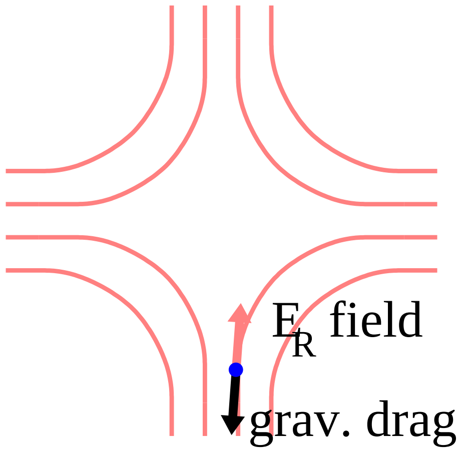


$$E_\rho{}^{\mathsf{a}}(t,r,\vartheta,\varphi) \quad \sim \quad \frac{1}{r\,\sin\vartheta}\sqrt{1-\frac{r_S}{r}}\begin{pmatrix} 0 \\ \sin\vartheta\,\left(1+\frac{r_S}{r}\,\ln(\frac{1}{2}\sin\vartheta)\right) \\ \frac{1}{r}\,\cos\vartheta \\ 0 \end{pmatrix},$$

Field of uniformly charged suspended wire.
(Asymptotically cylindric electric field, superimposed on a massive body.)

Earth-radial electrostatic field effectively exerted by beam focusing optics at beam trajectory:





$$E_R{}^{\mathsf{a}}(t,r,\vartheta,\varphi) \quad \sim \quad \frac{1}{r^2}\begin{pmatrix} 0 \\ \sqrt{1-\frac{r_S}{r}} \\ 0 \\ 0 \end{pmatrix}$$

Holds stationary beam against falling.

(Field of charged spherical shell around graviting body.)

Solution in the "dumbest" way

- *Class.Quant.Grav.***35**(2018)175003: just solve EOM to find $\beta,\ B_{\rm v},\ E_\rho,\ E_R$ providing
  – stationary planar circular beam,
  – spin precession in bending plane stopped ("Koop" condition).
- Due to high degree of symmetries, it can be solved even without approximation.
- Result expanded in terms of r_S:

$$\Omega_{\text{roll}} = \underbrace{0}_{\text{roll rate at } r_S \to 0 \text{ limit}} + -G\,\beta\gamma \underbrace{\frac{r_S\,c}{2\,R^2}}_{=\frac{g}{c}} + O({r_S}^2)$$

  Should be a benchmark result: any approximative method needs to reproduce it!

- A side-result:

$$q\,E_R = \underbrace{0}_{\text{"weight" of beam at } r_S \to 0 \text{ limit}} + m\,\gamma \underbrace{\frac{r_S\,c^2}{2\,R^2}}_{=g} + O({r_S}^2)$$

  [Nicely reproduces equivalence principle . . .]

Comparison and clarification

- *Phys.Rev.***D94**(2016)044019: do the same in lab frame, with perturbation in $\frac{r_S}{R}$.
(Can be more useful for real beam dynamics simulation, indeed.)
- In principle should agree with the results of manifestly covariant formalism.
- Claims:

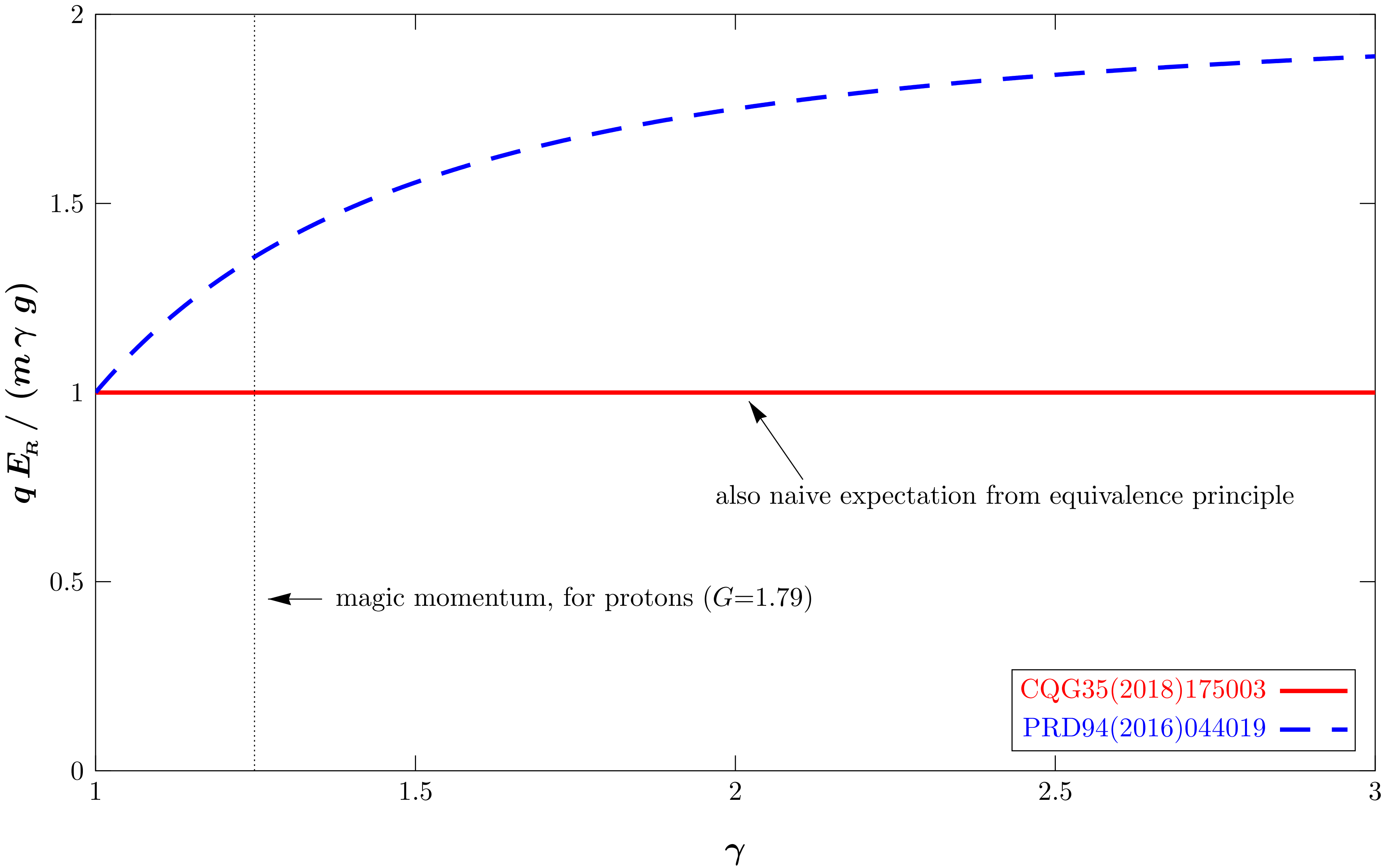
$$\Omega_{\text{roll}} = \underbrace{0}_{\text{roll rate at } r_S \to 0 \text{ limit}} + \frac{(1 - G\,(2\,\gamma^2 - 1))\,\beta}{\gamma} \underbrace{\frac{r_S\,c}{2\,R^2}}_{=\frac{g}{c}} + O({r_S}^2)$$

- Claim for the side-result on the electrostatic balance force:

$$q\,E_R = \underbrace{0}_{\text{“weight” of beam at } r_S \to 0 \text{ limit}} + m\,\frac{2\,\gamma^2 - 1}{\gamma} \underbrace{\frac{r_S\,c^2}{2\,R^2}}_{=g} + O({r_S}^2)$$

[A bit striking in terms of naive application of equivalence principle . . .]

The “weight” as a function of γ in the two calculations



[Indicates that something might be not fully OK.]

What explains the discrepancy?

Left hand side of Newton equation (four-acceleration) for stationary planar circular beam:

$$a^b = u^a \nabla_a u^b = \underbrace{\frac{\mathrm{d}u^b}{\mathrm{d}\tau}}_{=0} + u^a u^c \Gamma^b_{ac} = \begin{pmatrix} 0 \\ -\frac{\beta^2\gamma^2}{\rho} \sin\Theta \left(1 - \frac{r_S}{R}\right) + \gamma^2 \frac{r_S}{2R^2} \\ -\frac{\beta^2\gamma^2}{\rho} \frac{1}{R} \cos\Theta \\ 0 \end{pmatrix}$$

(ρ: beam bending radius, and we had $\rho \overset{!}{=} R \sin\Theta$ by coordinate convention)

Its Earth-radial metric projection:

$$-g_{ab}\,\hat{r}^a\, a^b = \underbrace{-\frac{\beta^2\gamma^2}{\rho} \sin\Theta}_{\substack{\text{constant part} \\ \text{(Earth-radial projection of} \\ \text{centrifugal four-acceleration} \\ \text{in Minkowski limit)}}} + \underbrace{(2\gamma^2 - 1)\,\frac{r_S}{2R^2}}_{\substack{\text{first order GR correction} \\ \text{as used in PRD94(2016)044019, correctly}}} + O({r_S}^2).$$

From this, one would think that indeed:

$$q\,E_R = m\,\frac{\mathbf{2\,\gamma^2 - 1}}{\boldsymbol{\gamma}}\,\underbrace{\frac{r_S\,c^2}{2\,R^2}}_{=g} + O({r_S}^2)$$

holds, as stated in PRD94(2016)044019.

But **Achtung**!

Also the right hand side of the Newton equation gets GR correction:

$$u^a \nabla_a u^b \;=\; -\frac{q}{m}\, \boldsymbol{g}^{\boldsymbol{bc}}\, F_{cd}\, u^d$$

$$\Downarrow$$

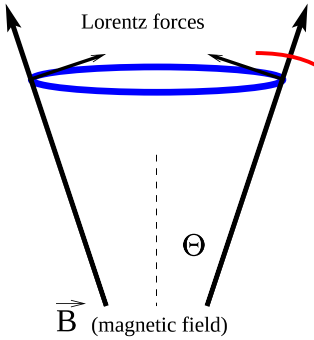
$$q\, E_R \;=\; m\, \boldsymbol{\gamma}\, \underbrace{\frac{r_S\, c^2}{2\, R^2}}_{=\,g} \;+\; O({r_S}^2)$$

if one solves Newton equation consistently, as stated in CQG35(2018)175003.

[Equivalence principle restored when considering **full** system . . .]

However!

Consider a magnetic bending field $B^a(t, r, \vartheta, \varphi)$ which is Earth-radial at the beam, not “vertical”.



$$B^a\Big|_{\text{at beam}} \quad \sim \quad \hat{r}^a, \qquad \text{then:}$$

$$-\frac{q}{m}\, g^{bc}\, F_{cd}\, u^d \quad = \quad -\frac{q}{m} \begin{pmatrix} 0 \\ -\,E_R\,\gamma\,\sqrt{1-\frac{r_S}{R}} \\ \frac{1}{R}\,(B\,\beta\gamma - E_\rho\,\gamma) \\ 0 \end{pmatrix}$$

$$\Downarrow$$

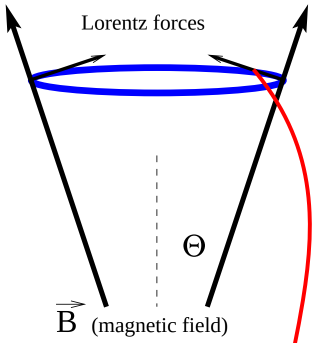
$$q\,E_R \quad = \quad \underbrace{-\,m\gamma\,\frac{\beta^2\,c^2}{\rho}\,\sin\Theta}_{\text{a constant offset in Minkowski limit from Earth-radial magfld}} \;+\; \underbrace{m\,\frac{\mathbf{2\gamma^2-1}}{\boldsymbol{\gamma}}\,\overbrace{\frac{r_S\,c^2}{2\,R^2}}^{=g}}_{\text{first order GR correction, as in PRD94(2016)044019}} \quad + \quad O({r_S}^2)$$

✓

a constant offset
in Minkowski limit
from Earth-radial magfld

first order GR correction,
as in PRD94(2016)044019

if one solves the general relativistic Newton equation consistently.



$$B^a\Big|_{\text{at beam}} \quad \sim \quad \hat{r}^a, \qquad \text{then:}$$

$$\Omega_{\text{roll}} \;=\; \underbrace{\frac{\beta\, c}{\rho}\,\frac{G\,\beta^2\gamma^2 - 1}{\gamma}\,\sin\Theta}_{\text{from beam-radial magfld component due to Earth-radial magfld assumption (can be very large effect!)}} \;+\; \underbrace{\frac{\mathbf{1 - G\,(2\gamma^2 - 1)}}{\boldsymbol{\gamma}}\,\boldsymbol{\beta}\,\overbrace{\frac{r_S\, c}{2R^2}}^{=\frac{g}{c}}}_{\text{first order GR correction, PRD94(2016)044019}} \;+\; O({r_S}^2)$$

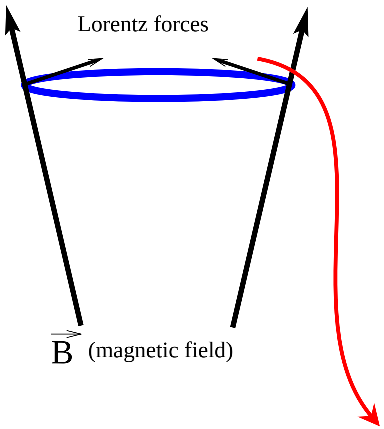
! ✓

from beam-radial magfld component
due to Earth-radial magfld assumption
(can be very large effect!)

first order GR correction,
PRD94(2016)044019

So, PRD94(2016)044019 is correct, but for Earth-radial magnetic bending axis!
(That however, is not frozen spin but a Koop “spin wheel” configuration, if not at magic mom.)

Good idea! What if we had *any* axisymm magnetic field shape imperfection in GR model ?



$$B^a\Big|_{\text{at beam}} \sim \underbrace{B_{\mathrm{v}}{}^a}_{\text{perfect}} + \underbrace{B_\rho{}^a}_{\text{imperfection}}, \quad \text{then:}$$

$$\Omega_{\text{roll}} = \underbrace{-\frac{q\,(1+G)}{m}\frac{1}{\gamma^2}B_\rho}_{\substack{\text{from magnetic}\\ \text{field imperfection}\\ \text{(same as Koop's formula)}}} + \underbrace{-G\,\beta\,\gamma\,\frac{\overbrace{r_S\,c}^{=\frac{g}{c}}}{2R^2}}_{\substack{\text{first order GR correction,}\\ \text{CQG35(2018)175003}}} + O({r_S}^2)$$

Special case:
$$B^a\Big|_{\text{at beam}} \sim \hat{r}^a$$
$$\Downarrow$$
$$\frac{B_\rho}{B_{\mathrm{v}}} = \tan\Theta\,\sqrt{1-\frac{r_S}{R}}$$

($\leftarrow$ B imperfection interferes with GR under Earth-radiality constraint !!)

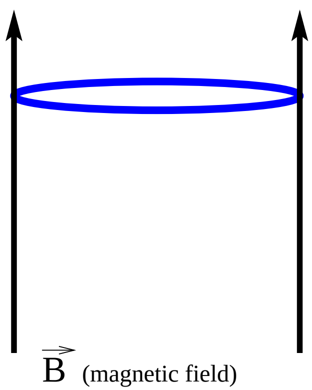
This also explains PRD94(2016)044019 !
(But again, it is for Koop spin wheel, not for frozen spin ring.)

Bottomline of the story:

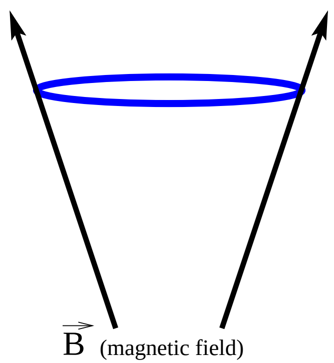
for perturbative treatment, there are **two** small parameters.

– GR correction $\frac{r_S}{R}$ ($\approx 10^{-9}$ for an experiment at the Earth surface),

– inclination of Earth-radial axis $\frac{\rho}{R}$ ($\approx 10^{-6}$ for an EDM ring vs Earth).



$$=$$



$$+ \quad O\left(\frac{\rho}{R}\right)$$

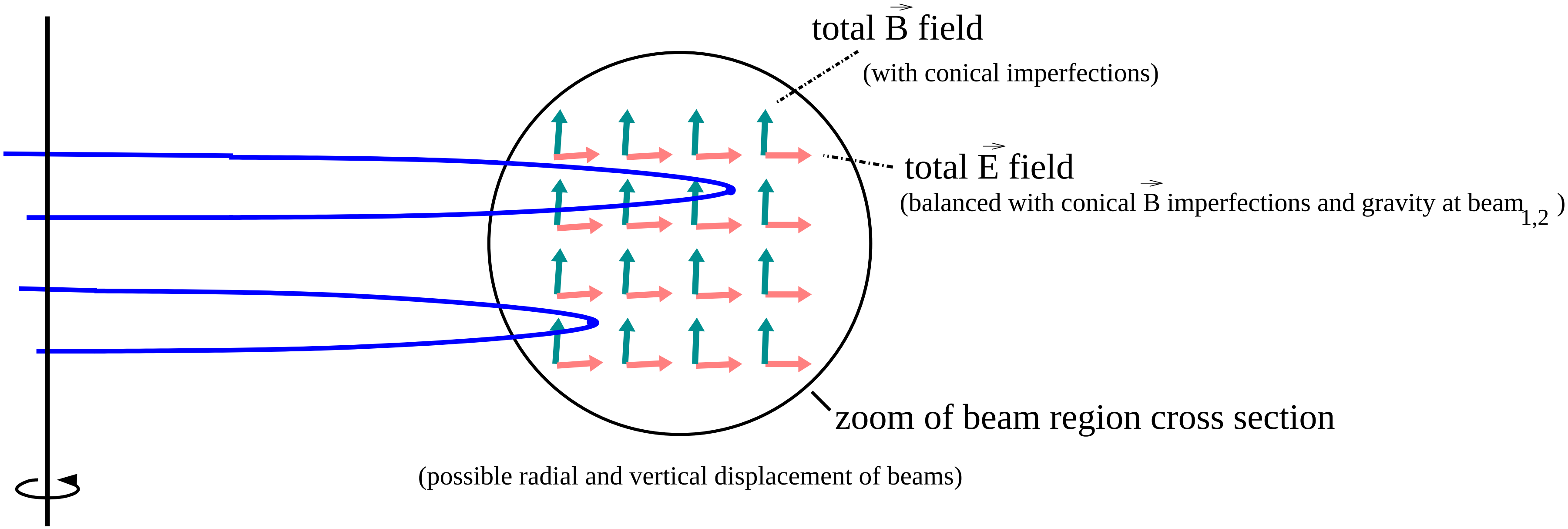
[If not considered, one neglects corresp magnetic field imperfections, which is a big effect.]

– Moreover: also RHS of Newton equation gets GR correction, not only LHS.

Reducing B_ρ systematics, arXiv2009.09820

See e.g. R.Talman `arXiv1812.05949`:

2 particle species in the same fields, both with frozen spin in the bending plane ("*a la* Koop").



Use

$$\delta\Omega_i^{\text{syst}} = -\frac{q_i\,(1+G_i)}{m_i}\,\frac{1}{{\gamma_i}^2}\,(B_\rho)_i \qquad (\text{ for beams } i = 1, 2\ )$$

to cancel B_ρ systematics in the combined signal $\Omega = \Omega_1 - W \cdot \Omega_2$.

[Due to Maxwell's equations, locally $B_\rho\,\rho = const \implies (B_\rho)_1\,\rho_1 = (B_\rho)_2\,\rho_2$]

- By the above, $W = \left(\frac{q_1\,(1+G_1)}{m_1} \, \frac{1}{{\gamma_1}^2} \Big/ \frac{q_2\,(1+G_2)}{m_2} \, \frac{1}{{\gamma_2}^2} \right) \frac{\rho_2}{\rho_1}$ is the optimal weighting.
- Actual W for the beams re-expressable as:

$$W = W\left(\frac{\mu_1}{\mu_2}, \frac{G_1}{G_2}, G_1, \frac{\rho_1}{\rho_2}\right),$$

(bottleneck: we cannot measure $\frac{\rho_1}{\rho_2}$ beyond 10^{-6} accuracy).

- Actual W for the beams re-expressable also as:

$$W = W\left(\frac{\mu_1}{\mu_2}, \frac{G_1}{G_2}, G_1, \boldsymbol{\frac{\omega_1}{\omega_2}}\right)$$

(much better accuracy for $\frac{\omega_1}{\omega_2}$, around 10^{-10} likely).

- For $p - {}^3he$ looks promising $(\rho = 10\,\mathrm{m})$:

E_ρ [kV/cm]	B_v [T]	p_1 [MeV/c]	p_2 [MeV/c]	$\Omega_{\mathrm{GR},1}$ [nrad/s]	$\Omega_{\mathrm{GR},2}$ [nrad/s]
-52.7	0.0276	271.9	-471.2	-17.0	-23.0

GR signal combines constructively:

combined Ω_{GR}	allowed $\langle \delta B_\rho \rangle$ for $\times 10$ GR / Syst
45.6 nrad/s	max $\frac{1}{\delta W}$ $2.3 \cdot 10^{-17}$ Tesla

error propagation favorable:

W	$\frac{\partial W}{\partial \mu_1/\mu_2}$	$\frac{\partial W}{\partial G_1/G_2}$	$\frac{\partial W}{\partial G_1}$	$\frac{\partial W}{\partial \omega_1/\omega_2}$
-1.24	2.04	3.03	-0.0093	0.71

Summary

- Contribution of $\langle \delta B_\rho \rangle$ + GR :

$$\Omega_{\text{roll}} = \underbrace{-\frac{q\,(1+G)}{m}\,\frac{1}{\gamma^2}\,\langle \delta B_\rho \rangle}_{magfield\ \ imperfection} + \underbrace{-\boldsymbol{G\,\beta\,\gamma}\,\frac{g}{c}}_{GR\ \ correction}$$

- They can interfere, so need to be handled with care.
- A “doubly-frozen” ring can remove $\langle \delta B_\rho \rangle$ term to first order (see R.Talman’s talk).
- For $p - {}^3\!he$ ring the GR signal combines constructively ($\approx 46\,\mathrm{nrad/sec}$).
- $\big|\,\langle \delta B_\rho \rangle\,\big| \leq 10^{-7}\,\mathrm{Tesla}$ may be enough for $\times 10$ GR/Syst.

(but only axisymmetric imperfections considered. . .)

Backup

Fundamental notations about the beam:

G : magnetic moment anomaly of the particle (also denoted by G in literature).

$m,\, q$: mass and charge of the particle.

$\beta\gamma$: momentum-over-mass of the particle in lab system.

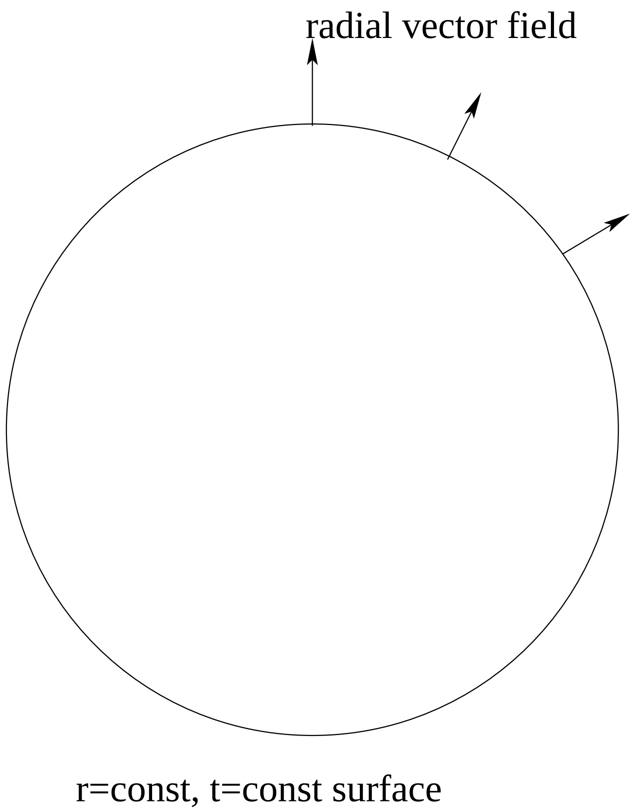
$\beta,\, \gamma$: velocity and Lorentz factor of the particle in lab system.

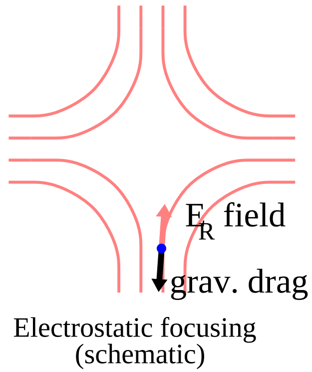
ρ : bending radius of the beamline.

$E_{\mathrm{v}},\, B_{\mathrm{v}}$: “vertical” electrostatic or magnetostatic field at the beamline.

$E_{\rho},\, B_{\rho}$: “beam-radial” electic or magnetic field at beamline.

Earth-radial electrostatic field effectively exerted by beam focusing optics:

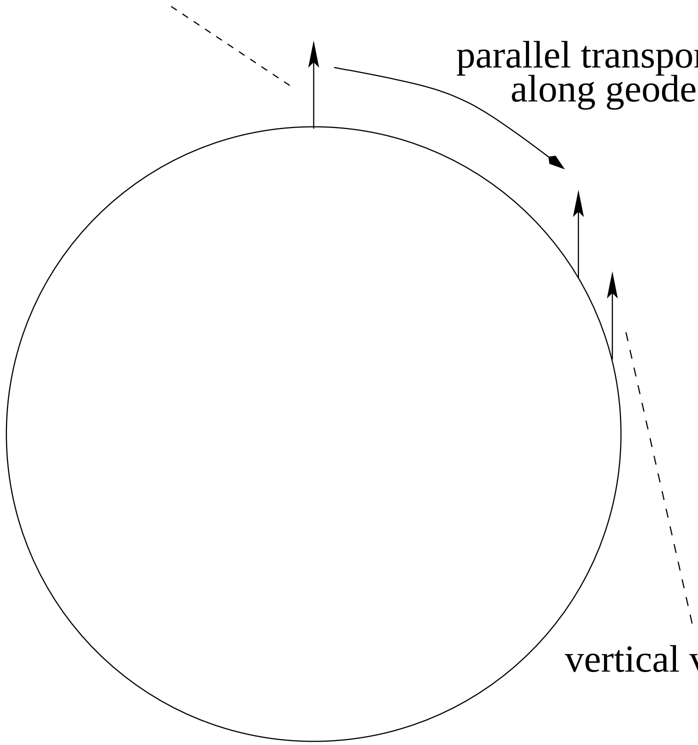




$$E_R{}^a(t,r,\vartheta,\varphi) = E_R\,\frac{R^2}{r^2}\begin{pmatrix} 0 \\ \sqrt{1-\frac{r_S}{r}} \\ 0 \\ 0 \end{pmatrix}$$

Holds the beam against falling. (Field of charged spherical shell around graviting body.)

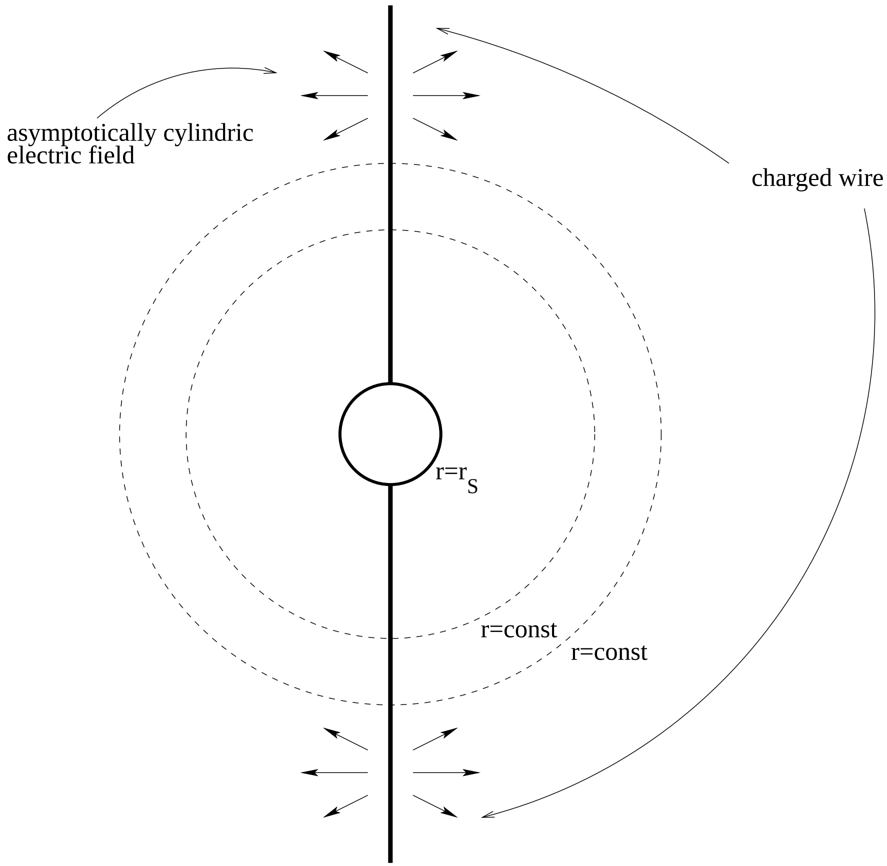
Vertical magnetic field (field inside infinite solenoid):



$$B^{\mathsf{a}}(t,r,\vartheta,\varphi) = B\sqrt{\frac{1-\frac{r_S}{r}}{1-\frac{r_S}{R}\left(\frac{\rho}{R}\right)^2}}\begin{pmatrix} 0 \\ \cos\vartheta \\ -\frac{1}{r}\sin\vartheta \\ 0 \end{pmatrix}$$

Bending field.

Beam-radial electrostatic field (field of infinite uniformly charged suspended wire):



$$E_{\rho}{}^{\mathsf{a}}(t,r,\vartheta,\varphi) = E_{\rho}\,\frac{\rho}{r\,\sin\vartheta}\,\sqrt{1-\frac{r_S}{r}}\,\mathcal{N}_{r_S}\begin{pmatrix} 0 \\ \sin\vartheta\,\left(1+\frac{r_S}{r}\,\ln(\frac{1}{2}\sin\vartheta)\right) \\ \frac{1}{r}\,\cos\vartheta \\ 0 \end{pmatrix},$$

Bending field.

$$\text{with } \mathcal{N}_{r_S} = \left(\left(\left(\frac{\rho}{R}\right)^2\left(1+\frac{r_S}{R}\ln\left(\frac{\rho}{2R}\right)\right)\right)^2 + \left(1-\left(\frac{\rho}{R}\right)^2\right)\left(1-\frac{r_S}{R}\right)\right)^{-\frac{1}{2}}$$

We have:

– spacetime metric $g_{\mathsf{ab}}(t, r, \vartheta, \varphi)$,

– magnetic bending $B_{\mathrm{v}}{}^{\mathsf{a}}(t, r, \vartheta, \varphi)$,

– electrostatic bending $E_{\rho}{}^{\mathsf{a}}(t, r, \vartheta, \varphi)$,

– Earth-radial electrostatic field $E_{R}{}^{\mathsf{a}}(t, r, \vartheta, \varphi)$ (exerted by focusing, keeping from fall),

– ansatz of planar circular movement at $r = const$, $\vartheta = const$ with velocity β,

and then we find the β, B_{v}, E_{ρ}, E_{R} settings, for the equation of motion to be satisfied:

$$u^a \nabla_a u^b = -\frac{q}{m} F^{bc} u_c,$$

$$D_u^F s^b = -(G+1) \frac{q}{m} \left(F^{bc} - u^b u_d \; F^{dc} - \; F^{bd} u_d u^c \right) s_c$$

Beam evolution equations in lab frame

(Newton + Thomas-Bargmann-Michel-Telegdi equations):

$$\frac{\mathrm{d}\vec{\beta}}{\mathrm{d}t_{\text{lab}}} = \frac{q}{m\,\gamma}\left(\vec{E} - (\vec{\beta}\cdot\vec{E})\,\vec{\beta} + \vec{\beta}\times\vec{B}\right),$$

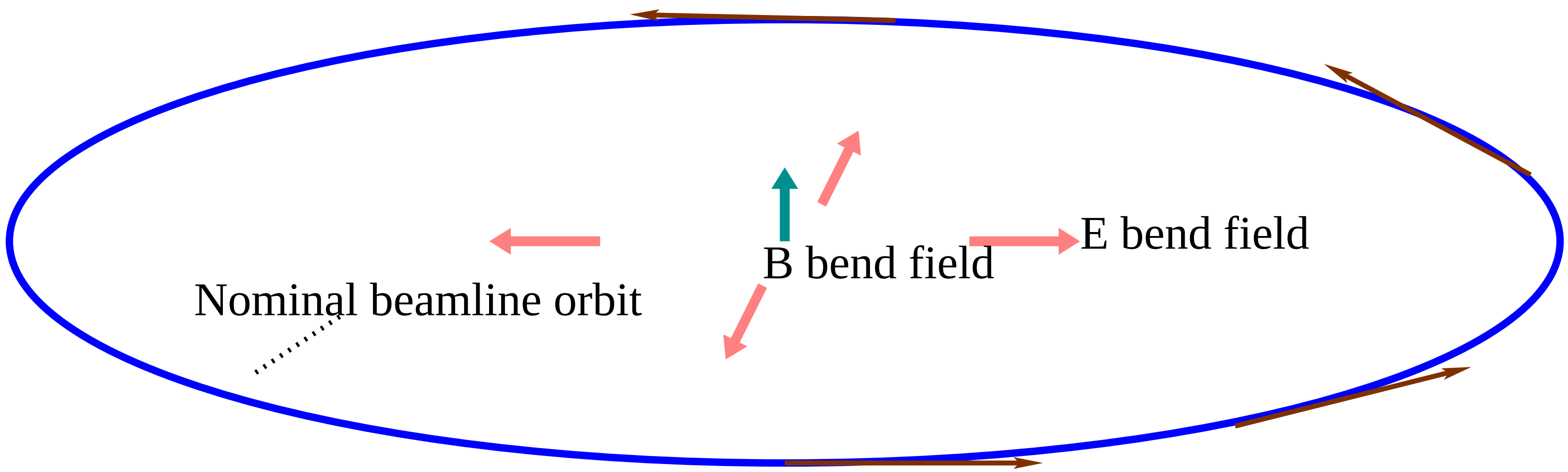
$$\frac{\mathrm{d}\vec{S}_{\text{lab,corot.}}}{\mathrm{d}t_{\text{lab}}} = -\frac{q}{m}\left(\underbrace{G\,\vec{B}}_{\text{magnetic term}} + \underbrace{\overbrace{\left(\frac{1}{(\beta\gamma)^2} - G\right)}^{\text{zero at “magic momentum”}}\vec{\beta}\times\vec{E}}_{\text{electric term}} + \underbrace{\frac{1}{2}\,\eta\,\left(\vec{E}+\vec{\beta}\times\vec{B}\right)}_{\text{EDM term}}\right)\times\vec{S}_{\text{lab,corot.}}$$

($\mathrm{g} = \frac{2\,m\,\mu}{q\,S}$, $G = \frac{\mathrm{g}-2}{2}$ is magnetic moment anomaly, $\eta = \frac{2\,m\,d}{q\,S}$ is the “g” of EDM.)

Magic momentum: $|\beta\gamma| = \frac{1}{\sqrt{G}}$ (only possible for $G > 0$).

Frozen spin condition: $\frac{\mathrm{d}\vec{S}_{\text{lab,corot.}}}{\mathrm{d}t_{\text{lab}}} = 0$ (assuming $\eta = 0$).

(Special case: with $\vec{B} = 0$, then magic momentum $\Leftrightarrow$ frozen spin condition.)

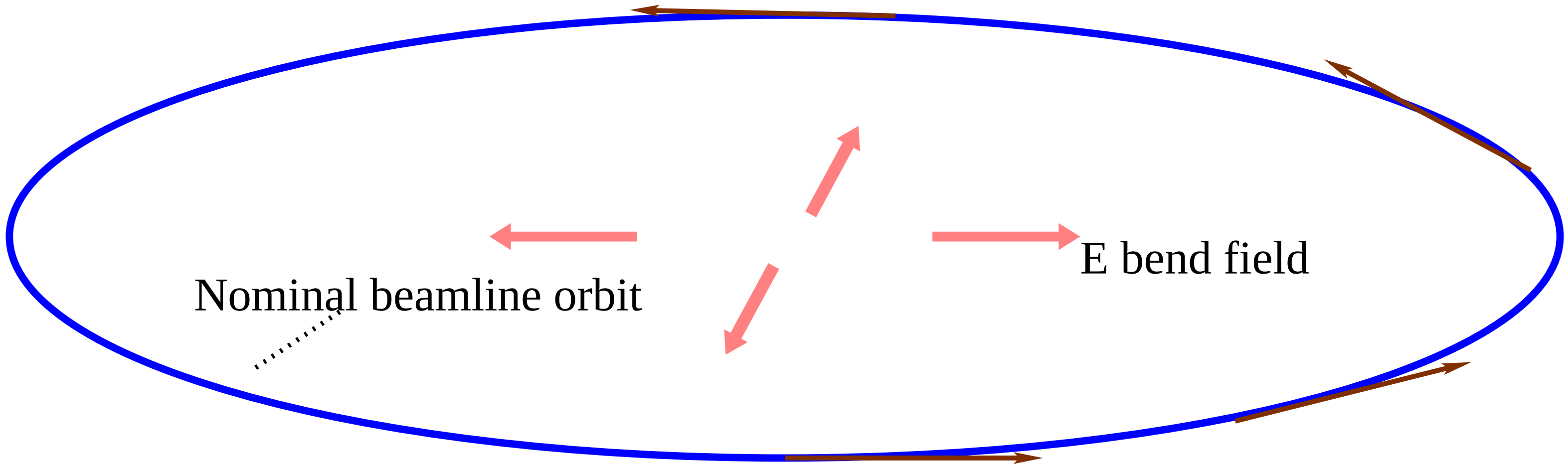


Longitudinally polarized beam circulates in a magnetic + electric storage ring, such that:
– planar circular beam orbit is satisfied,
– spin precession due to magnetic moment anomaly is stopped.

Then, EDM would torque the spin around instantaneous beam-radial axis, spin would “roll”.

(***frozen spin storage ring***, see e.g. Semertzidis *et al*: *PRL* **93**(2004)052001)

A well known extreme case:

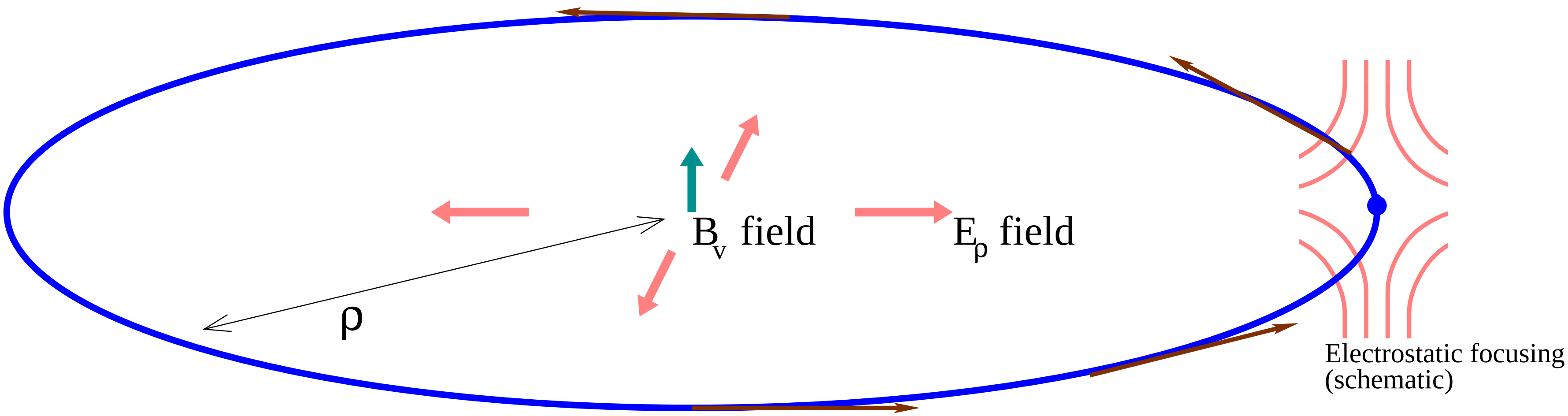


“Electrostatic-only” or “magic momentum” frozen spin ring.

(When $\vec{B} = 0$, then frozen spin condition $\Longleftrightarrow$ magic momentum.)

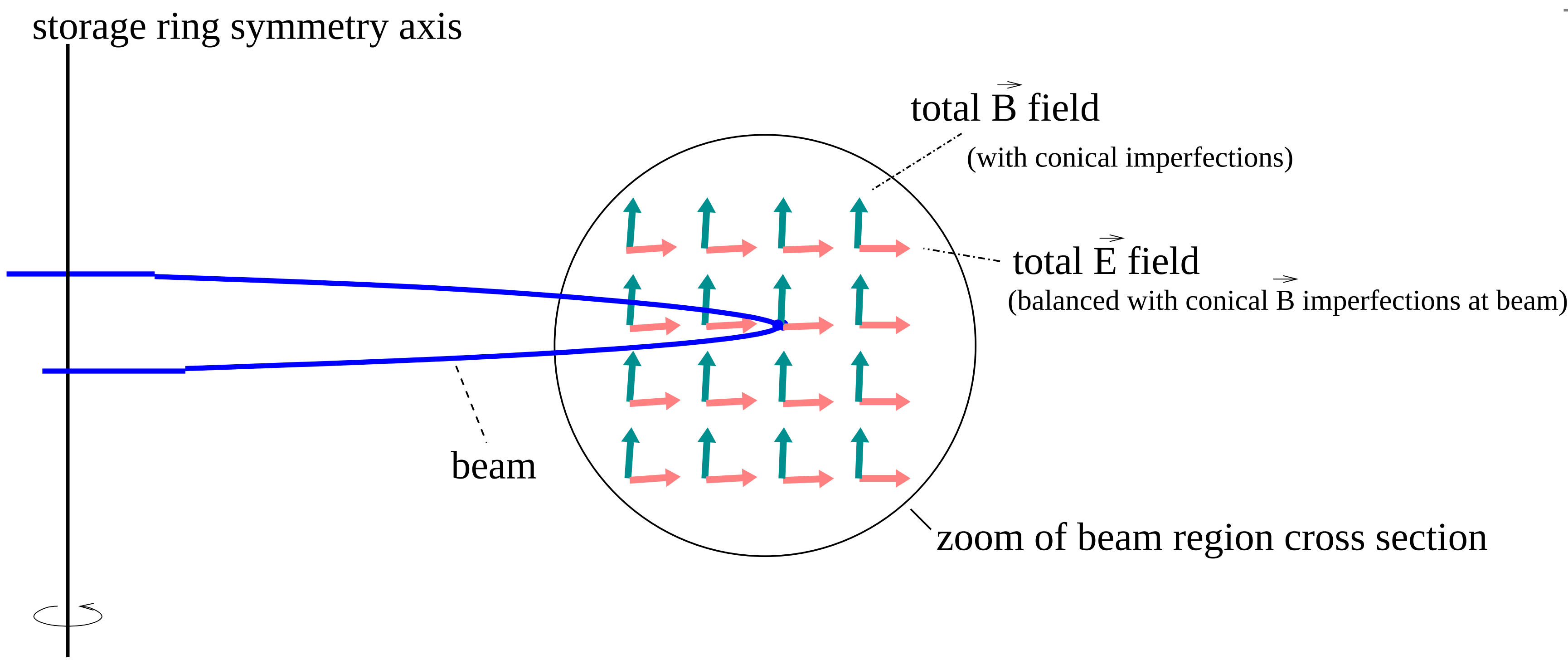
Only possible for $G > 0$ particles, not possible for d, ^3he, $\ldots$

In reality, a transverse electrostatic quad (or higher-pole) field is also applied for focusing:

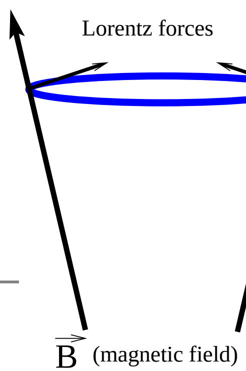


[Idealized closed planar circular beam passes at zero field in focusing optics, so not affected.]

Close look when B_ρ is present:



E.g. when E focusing is used, and B imperfection present (but imperfection is axisymmetric).

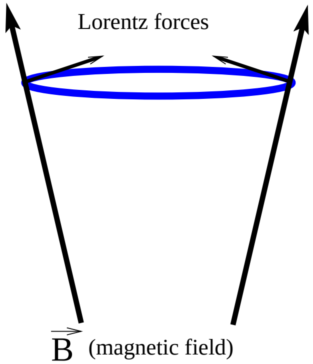


(total $\vec{B}$ field is "cone/bucket shaped" at nominal beam trajectory)

⬇

we may call presence of B_ρ the "axisymm imperfection" of $\vec{B}$ shape

When B_ρ is present in frozen spin / Koop ring with E focusing:



Upward Lorentz force by B_ρ keeps balance with E_v force by focusing.

(i.e. nominal stationary beam does not drift vertically)

Total "roll" frequency because of B_ρ is the most important EDM systematics:

$$\delta\Omega \quad = \quad \underbrace{-\frac{q\,(1+G)}{m}\,\frac{1}{\gamma^2}\,B_\rho}_{\text{magnetic field axisymm imperfection term}}$$

Comes from:

– explicit torque by B_ρ around instantaneous beam-radial axis,

– additional torque by E_v from focusing (vanishes at magic momentum).

(easy to derive, but see also I.Koop: *Proc.IPAC2013*(2013)TUPWO040)

Some philosophy…

To what extent it is gravitational modification of kinematics vs Larmor precession?

$$-\overbrace{\frac{\mathrm{g}-2}{2}}^{G}\;\beta\gamma\;\; g/c$$

(← vertical polarization buildup rate by GR)

$$\underbrace{D_u^F s^b}_{\text{from kinematics (Thomas)}} = \underbrace{-\mathrm{g}\,\frac{q}{2\,m}\left(F^{bc} - u^b u_d\; F^{dc} - \;F^{bd}\,u_d u^c\right)\, s_c}_{\text{of electrodynamic origin (Larmor)}}$$

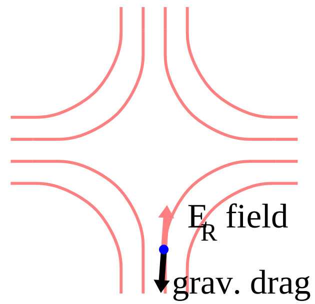
(← TBMT equations)

$$\left(\quad F_{ab} = \underbrace{F^B_{ab}}_{\text{magnetic bending}} + \underbrace{F^{E_\rho}_{ab}}_{\text{electric bending}} + \underbrace{F^{E_R}_{ab}}_{\text{just compensating “weight” of beam}} \quad \right)$$

Part of the contribution is coming merely from “weight”:

$$-\frac{\mu}{S}\left(F_{bc}^{E_R} - u_b u^d\, F_{dc}^{E_R} - F_{bd}^{E_R}\, u^d u_c\right) \qquad (\leftarrow \text{ Larmor precession by } E_R)$$

E_R merely compensates the gravitational drag of Earth:



Kind of “classical” effect. What is the contribution of Larmor precession by E_R ?
Answer:

full GR prediction : E_R contribution $= \quad G : (1+G)$

Also:

full GR prediction : semi-classical prediction $= \quad G : (1+G)$

Some historical terminology on the free gyroscope equation $D_u^F s^b = 0$:

- For free, i.e. geodesic motion ($u^a \nabla_a u^b = 0$) in gravitational field:
  - In the field of nonrotating object (Schwarzschild): *de Sitter* precession.
  - In the field of rotating object (Kerr): *Lense-Thirring* precession.
    (See also: *Gravity Probe B* satellite experiment.)

- For forced orbit ($u^a \nabla_a u^b =$ some force):
  - It is called the *Thomas* precession.
    (Already gives effect in special relativity, i.e. in absence of gravity.)

Our case:

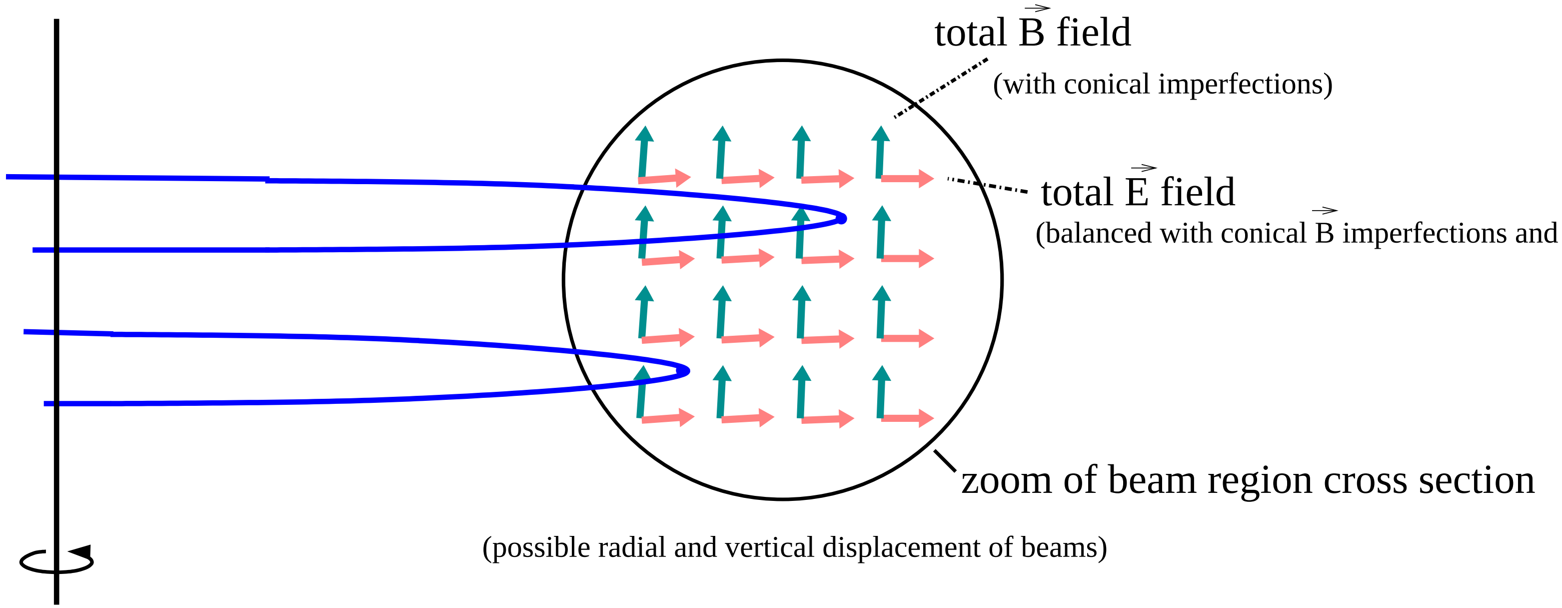
$$\underbrace{D_u^F s^b}_{\text{causes Thomas precession}} = \underbrace{-\frac{\mu}{S}\left(F^{bc} - u^b u_d\, F^{dc} - F^{bd}\, u_d u^c\right) s_c}_{\text{causes Larmor precession in addition}}$$

Both of them suffer GR corrections.

Reducing B_ρ systematics, arXiv2009.09820

See e.g. R.Talman arXiv1812.05949:

2 particle species in the same fields, both with frozen spin in the bending plane ("*a la* Koop").



Then, use $$\delta\Omega_i^{\text{syst}} = -\frac{q_i\,(1+G_i)}{m_i}\,\frac{1}{{\gamma_i}^2}\,(B_\rho)_i \qquad (\text{ for beams } i = 1, 2\ )$$

and the Maxwell's equations

$$\begin{aligned} B_\rho\,\rho &= const &:&\quad (B_\rho)_1\,\rho_1 = (B_\rho)_2\,\rho_2, \\ E_\rho\,\rho &= const &:&\quad (E_\rho)_1\,\rho_1 = (E_\rho)_2\,\rho_2, \\ B_\text{v} &= const &:&\quad (B_\text{v})_1 = (B_\text{v})_2 \end{aligned}$$

to cancel B_ρ systematics in the combined signal $\Omega = \Omega_1 - W \cdot \Omega_2$.

Kinematic equations for “Koop condition”:

$$(E_\rho\,\rho) \;=\; (E_{\rho\,i}\,\rho_i) \;=\; -\frac{m_i\,c^2}{q_i}\,\frac{G_i}{1+G_i}\,\beta_i^2\gamma_i^3,$$

$$B_{\rm v}\;c\;\rho_i \;=\; \frac{m_i\,c^2}{q_i}\,\frac{G_i}{1+G_i}\;(\beta_i\gamma_i)\left(\frac{1}{G_i}-(\beta_i\gamma_i)^2\right)$$

Given a fixed ring setting $\Rightarrow$ $(E_\rho\,\rho)$ is given $\Rightarrow$ β_i given $\Rightarrow$ $(B_{\rm v}\,\rho_i)$ given.

– We establish E_ρ, β_1 and $B_{\rm v}\,\rho_1$ (scan).
– Then we establish β_2 and ρ_2 (scan). [We aim for $\rho_2 \approx \rho_1$ in practice.]
– After, we need to know the precise value of W in this actually established setting.

– It is possible to express the true W as a function:

$$W \;=\; W\big(\tfrac{\mu_1}{\mu_2},\,\tfrac{G_1}{G_2},\,G_1,\,\tfrac{\rho_1}{\rho_2}\big),$$

but technically the bottleneck is that we cannot measure $\frac{\rho_1}{\rho_2}$ beyond 10^{-6} accuracy.

– It is also possible to re-express the true W as a function:

$$W \;=\; W\big(\tfrac{\mu_1}{\mu_2},\,\tfrac{G_1}{G_2},\,G_1,\,\tfrac{\omega_1}{\omega_2}\big),$$

much better accuracy for $\frac{\omega_1}{\omega_2}$, around 10^{-10} likely.

- Kinematic equation for “Koop condition”
(planar circular motion and frozen spin in bending plane):

$$
\begin{aligned}
E_\rho\,\rho &= -\,\mathrm{sign}(G)\,\frac{m\,c^2}{q}\,\frac{(G\,\beta\gamma)^2\sqrt{G^2+(G\,\beta\gamma)^2}}{G^2\,(1+G)}, \\
B_\mathrm{v}\,\rho &= \frac{m\,c}{q}\,\frac{(G\,\beta\gamma)(G-(G\,\beta\gamma)^2)}{G^2\,(1+G)}
\end{aligned}
$$

- Observe:
The necessary $|E_\rho|$ grows monotonically as $\sim|G\,\beta\gamma|^3$, for large $|G\,\beta\gamma|$.
The necessary $|E_\rho|$ decreases as $\sim|G|^{-2}$, for large $|G|$.

Experimental limitation is in $|E_\rho|$: above $8\,\mathrm{MV/m}$, essentially impossible.

- Experimental idea:
Use large $|G|$ particle (nucleus), so that too large $|E_\rho|$ can be avoided.

Experimental / financial constraints:

ring radius ρ maximum $\sim 10\,\mathrm{m}$,

magnetic field $|B_\mathrm{v}|$ maximum $\sim 1\,\mathrm{Tesla}$,

electric field $|E_\rho|$ maximum $\sim 8\,\mathrm{MV/m}$.

Let us aim for a GR signal strength $|G\,\beta\gamma| = 0.4 \quad (13.1\,\mathrm{nrad/sec})$.
Assume a surely realistic electric field $|E_\rho| = 4.10\,\mathrm{MV/m}$.

Possible settings:

particle	$G\ (\approx)$	ρ [m]	$\lvert B_\mathrm{v}\rvert$ [Tesla]	p [MeV/c]	$\mathcal{E}_\mathrm{kin}$ [MeV]
triton	7.92	**1.55**	0.0335	141.9	3.58
helion3	-4.18	**4.13**	0.0353	268.5	12.8
proton	1.79	**7.50**	0.0304	209.7	23.1

Not realistic settings:

particle	$G\ (\approx)$	ρ [m]	$\lvert B_\mathrm{v}\rvert$ [Tesla]	p [GeV/c]	$\mathcal{E}_\mathrm{kin}$ [GeV]
deuteron	-0.142	1796	0.0243	5.283	3.731
electron	0.00116	5942	0.0136	0.1765	0.1760
muon	0.00116	1228520	0.0136	36.497	36.391

Some GR

Stationary planar circular beam orbit:

$$\boldsymbol{\gamma}_\omega(t) = \begin{pmatrix} t \\ R \\ \Theta \\ \omega\sqrt{1-\frac{r_S}{R}}\,t \bmod 2\pi \end{pmatrix}, \qquad \dot{\boldsymbol{\gamma}}_\omega{}^{\mathsf{a}}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \omega\sqrt{1-\frac{r_S}{R}} \end{pmatrix},$$

$$u_\omega{}^{\mathsf{a}} = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{r_S}{R}}}\gamma \\ 0 \\ 0 \\ \omega\gamma \end{pmatrix} \qquad (\rho = R\sin\Theta, \quad \beta = \rho\,\omega, \quad \gamma = \tfrac{1}{\sqrt{1-\beta^2}}).$$

$$a_\omega{}^{\mathsf{b}} = u_\omega{}^{\mathsf{a}}\nabla_{\mathsf{a}} u_\omega{}^{\mathsf{b}} = \underbrace{\frac{\mathrm{d}u_\omega{}^{\mathsf{b}}}{\mathrm{d}\tau}}_{=0} + u_\omega{}^{\mathsf{a}} u_\omega{}^{\mathsf{c}} \Gamma^{\mathsf{b}}_{\mathsf{ac}} = \begin{pmatrix} 0 \\ -\frac{\beta^2\gamma^2}{\rho}\sin\Theta\left(1-\frac{r_S}{R}\right) + \gamma^2\frac{r_S}{2R^2} \\ -\frac{\beta^2\gamma^2}{\rho}\frac{1}{R}\cos\Theta \\ 0 \end{pmatrix}$$

Polarimeter (observer):

$$t \mapsto \boldsymbol{\gamma}_{0,\phi}(t) = \begin{pmatrix} t \\ R \\ \Theta \\ \phi \end{pmatrix}, \qquad \dot{\boldsymbol{\gamma}}_0{}^{\mathsf{a}}(t, R, \Theta, \varphi) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$u_0{}^{\mathsf{a}} = \begin{pmatrix} \frac{1}{\sqrt{1 - \frac{r_S}{r}}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Focusing vs imperfection

Imposing:
– stationary planar circular beam,
– spin precession due to magnetic moment in bending plane stopped (“Koop condition”)
Then, it selects uniquely E_ρ and B_v, for given β.
B_ρ and E_v keep the beam in vertical balance, so they determine each-other, given β.
Either B_ρ or E_v or their combination is a free parameter.
It determines the roll rate by Thomas+Larmor (+GR) effect.

Hybrid ring

Semertzidis et al (see also *Phys.Rev.Accel.Beams* **22**(2019)034001):
E-only (magic mom.) ring, but B-focusing.
Then, $\langle \delta B_\rho \rangle$ imperfection is taken care of by focusing. But then $\langle \delta E_\mathrm{v} \rangle$ must be taken care of.

GR prediction *a la* CQG35(2018)175003: $\Omega^{\mathrm{GR}}_{\mathrm{roll}} = -\,\mathrm{sign}(\beta\gamma)\sqrt{G}(G+2)\,\frac{g}{c} \approx \mp 167$ nrad/s.

GR prediction *a la* PRD94(2016)044019: $\Omega^{\mathrm{GR}}_{\mathrm{roll}} = -\,\mathrm{sign}(\beta\gamma)\sqrt{G}(G+3)\,\frac{g}{c} \approx \mp 212$ nrad/s.
(again, difference comes from difference in grav.drag of $m\gamma\, g$ vs $m\frac{2\gamma^2-1}{\gamma}\, g$)

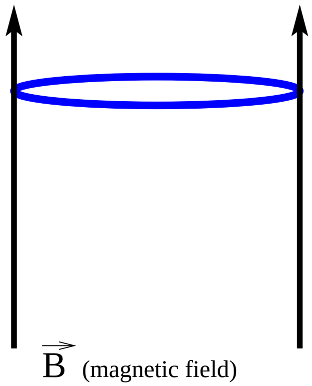
$\langle \delta E_\mathrm{v} \rangle$ contribution: $\Omega^{\mathrm{Syst}}_{\mathrm{roll}} = -\frac{q\,(G+1)}{m\,c\,\beta\,\gamma^2}\;\langle \delta E_\mathrm{v} \rangle$.

For single beam: GR/Syst≥10 $\iff \left| \langle \delta E_\mathrm{v} \rangle \right| \leq 10^{-16}\,|E_\rho|$ for 10m ring.

For counter-rotating beams: both imperfection and GR cancels altogether.

A testbench for doubly-frozen ring concept

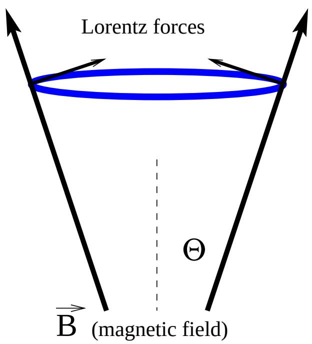
Take either:



(perfect)

$$\Omega_{\text{roll}} = \underbrace{0}_{\text{roll rate at } r_S \to 0 \text{ limit}} + \underbrace{-G\,\beta\gamma\, \overbrace{\frac{r_S\;c}{2\,R^2}}^{=\frac{g}{c}}}_{\text{first order GR correction, CQG35(2018)175003}} + O({r_S}^2)$$

Or take:



(Earth-radial)

$$\Omega_{\text{roll}} = \underbrace{\frac{\beta\,c}{\rho}\,\frac{G\,\beta^2\gamma^2 - 1}{\gamma}\,\sin\Theta}_{\text{roll rate at } r_S \to 0 \text{ limit}} + \underbrace{\frac{1 - G\,(2\gamma^2 - 1)}{\gamma}\,\beta\,\overbrace{\frac{r_S\;c}{2R^2}}^{=\frac{g}{c}}}_{\text{first order GR correction, PRD94(2016)044019}} + O({r_S}^2)$$

With a doubly-frozen spin ring: combined roll rate does not depend on above magfld models.