Spin Evolution and Decoherence

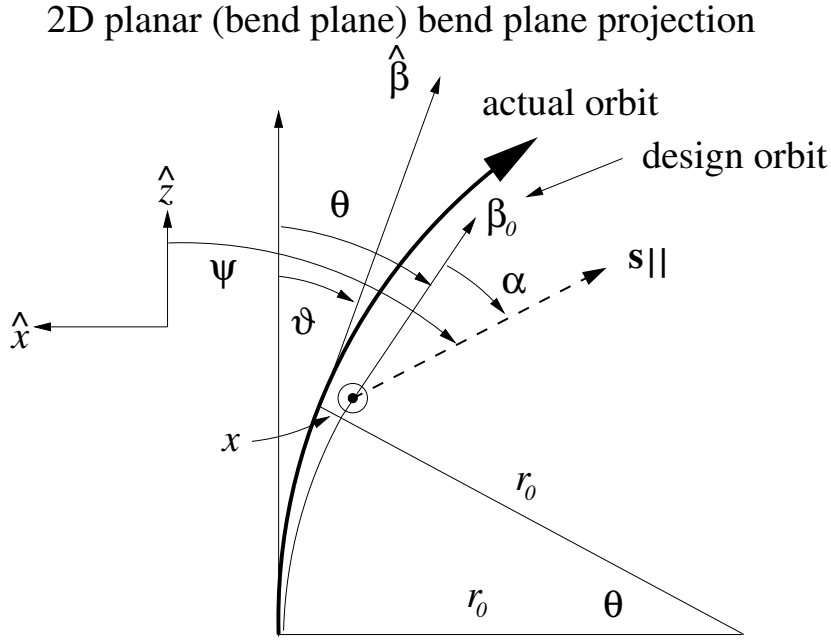
Richard Talman
Laboratory for Elementary-Particle Physics
Cornell University

22 January, 2018, Juelich

2 Outline

- 2D spin evolution through bends
  - Spin tune with superimposed E and B fields
  - Frozen spin condition with superimposed E and M fields
  - Global evolution of spin vector $\mathbf{s}_{\|}$ in E+M field
  - Solving the BMT equation in pure electric field
  - Re-derivation to include horizontal betatron displacement
- Virial theorem decoherence calculation
- Spinor evolution in idealized lattice
  - Spinor formalism
  - Spinor representation of spin evolution
- Storage ring as "Penning-Like Trap"
  - Motivation
  - Small deviations from magic condition

3 Spin evolution, approximately horizontal orbits



- Projection of 3D orbit onto the 2D laboratory horizontal design plane. x is the deviation of the (bold face) particle orbit from the (pale face) design orbit.
- If the bend plane coincides with the design bend plane (as is always approximately the case) $\hat{\boldsymbol{\beta}}_0$ and $\hat{\mathbf{z}}$ are identical.
- θ is the reference particle global horizontal angle and ϑ is the tracked particle global horizontal angle .
- Betatron oscillations cause θ and ϑ to differ (slightly) on a turn by turn basis but, on the average, they are the same.
- ψ is the global orientation angle of spin vector $\mathbf{s}_{parallel}$.
- α is the angle between $\mathbf{s}_{parallel}$ and the particle orbit.

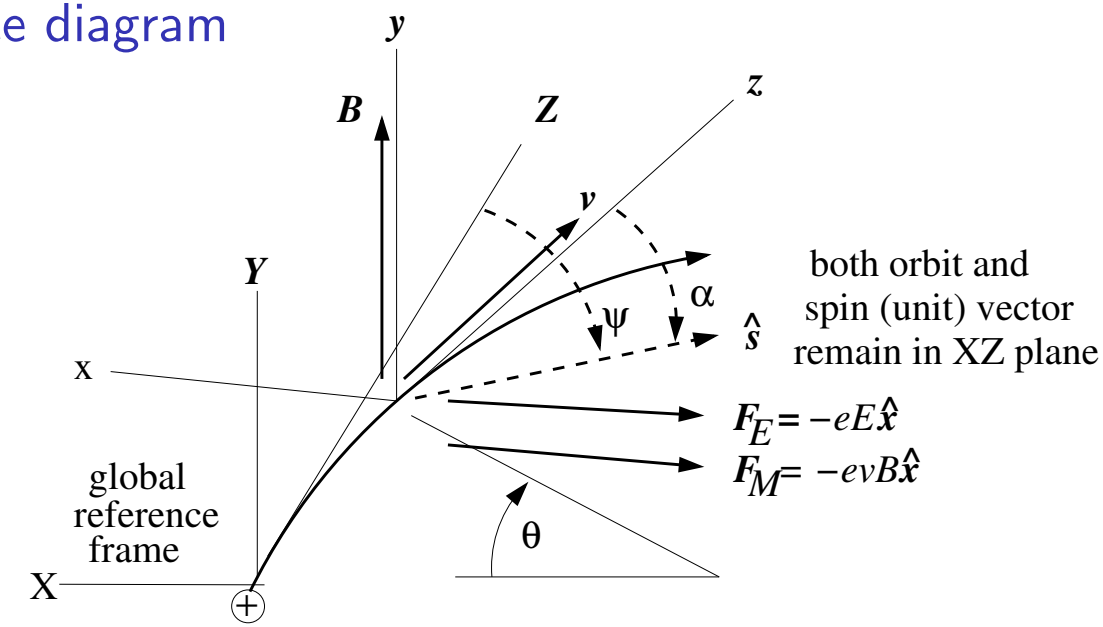
4 2D spin precession during circular motion

- For motion restricted to a single plane the BMT equation can be solved exactly in closed form.
- In this frame any precession of the spin is purely around an axis normal to the plane. Because of ultraweak vertical focusing, (e.g. in the WW-AG-CF lattice) vertical betatron oscillations are negligible for the 2D evolution through electric bend elements. Any betatron oscillations actually present are treated as exactly horizontal.
- The initial spin vector is

$$\mathbf{s} = -s_{\|} \sin\psi_0 \, \hat{\mathbf{X}} + s_y \hat{\mathbf{Y}} + s_{\|} \cos\psi_0 \, \hat{\mathbf{Z}}. \tag{1}$$

Here $s_y \hat{\mathbf{Y}}$ is the out-of-plane component of $\mathbf{s}$, $s_{\|}$ is the magnitude of the in-plane projection of $\mathbf{s}$, and α is the angle between the projection of $\mathbf{s}$ onto the plane and the tangent vector to the orbit.

5 Vector force diagram



2D circular motion in combined electric and magnetic field

- For a positive particle moving away, along the positive-z axis, with increasing global angle θ, for electric field $\mathbf{E} = -E\hat{\mathbf{x}}$ and magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$ to sum constructively, causing the particle to veer to the right (in the negative-x direction), requires both E and B to be positive.
- For positive spin tune Q_s the spin precession angle α increases with increasing θ; i.e.

$$\frac{d\alpha}{d\theta} = Q_s. \tag{2}$$

6 Spin tune with superimposed E and B fields

- Consider the precession of (unit magnitude) spin vector **s** lying in the (x, z) plane and belonging to a proton moving in the same (x, z) plane,
- For the electric $-E\hat{\mathbf{x}}$ and magnetic $B\hat{\mathbf{z}}$ fields shown in the figure, Jackson (11.171) (in MKS units) gives the time rate of change

$$\begin{aligned}
\frac{d}{dt}(\hat{\beta}\cdot\mathbf{s}) &= -\sin\alpha\,\frac{d\alpha}{dt} \\
&= -\frac{1}{mc/e}\,\mathbf{s}\cdot\left(\left(\frac{g}{2}-1\right)(\hat{\beta}\times\mathbf{cB})+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)(-\hat{\mathbf{x}}E)\right) \\
&= -\frac{1}{mc/e}\,\frac{1}{\gamma\beta}\left(G\gamma\beta cB+\left(G\gamma-\frac{1+G}{\gamma}\right)E\right)\,\sin\alpha \\
&= -\frac{1}{pc/e}\left(Q_s^M\beta cB+Q_s^E E\right)\,\sin\alpha
\end{aligned}$$

where $G=\dfrac{g}{2}-1,\ Q_s^M=G\gamma,$ and $Q_s^E=G\gamma-\dfrac{1+G}{\gamma}.$ (3)

7 Assuming the orbit is circular with radius r_0, the Lorentz law provides the centripetal force such that

$$\frac{v}{r_0} = \frac{d\theta}{dt} = \frac{\beta cB + E}{p/e}. \tag{4}$$

Combining the two previous equations produces

$$\frac{d\alpha}{d\theta} = Q_s = \frac{Q_s^M \beta cB + Q_s^E E}{\beta cB + E} = \eta_M Q_s^M + \eta_E Q_s^E, \tag{5}$$

where "fractional bend coefficients",

$$\eta_M = \frac{\beta cB}{\beta cB + E}, \quad \eta_E = \frac{E}{\beta cB + E}, \tag{6}$$

satisfying $\eta_M + \eta_E = 1$, have been introduced. As well as deriving the spin tune formulas in pure E and B fields, this has shown, when the bending field is a superposition of electric and magnetic fields, that the spin tune fractions are equal to the bend fractions.

8 Superimposed electric and magnetic bending—protons

We require the resulting spin tune Q_{EM} to vanish;

$$Q_{EM} = \eta_E Q_E + (1 - \eta_E) Q_M = 0. \tag{7}$$

Solving for η_E,

$$\eta_E = \frac{G}{G+1} \gamma^2. \tag{8}$$

For example, try $\gamma = 1.25$;

$$\eta_E = \frac{1.7926}{2.7926} \times 1.25^2 = 1.000, \tag{9}$$

which agrees with the "magic" proton value, for which no magnetic bending is required. In the non-relativistic limit $\gamma = 1$ and

$$\eta_E^{\mathrm{NR}} = \frac{1.7926}{2.7926} = 0.6419 \approx \frac{2}{3}. \tag{10}$$

9 Superimposed electric and magnetic bending—deuterons

- For the deuteron $G_d := -0.1429872721$. Expressed in terms of η_E, the frozen spin condition is

$$\eta_E = \frac{G}{G+1}\gamma^2 = \frac{-0.1429872721}{1-0.1429872721}\gamma^2 \tag{11}$$

- Because $\gamma^2 > 0$, this condition requires η_E to be negative.
- For example, with $r_0 = 8.4508$ and $\gamma_d = 1.13325$, (conditions assumed in the 2009 BNL proposal) $\eta_E = -0.21427$ and $\eta_M = 1.21427$. The resulting magnetic and electric fields are

$$B = 0.47929\,\mathrm{T}, \quad E = -53.9\,MV/m. \tag{12}$$

- Though B agrees almost exactly with the BNL proposal[4], the electric field disagrees by a large (and unattainable) factor.
- Perhaps the disagreement results from my formalism's requiring the electric and magnetic fields to "fight each other"? Clearly this disagreement needs to be resolved!!!

10 Global evolution of 2D spin vector $\mathbf{s}_{\parallel}$ in E+M field

- One motivation for this section is to investigate the disagreement just mentioned concerning superimposed electric and magnetic bending.
- Previously it is the evolution of the local angle α giving the spin orientation relative to the orbit that has been used to calculate the spin tune.
- Here, instead, we evolve the 2D spin vector $\mathbf{s}_{\parallel}$ itself, without reference to its orientation relative to the particle direction.
- Since the spin vector has unit length and lies in a horizontal plane, it is sufficient to keep track of a single global angle ψ.
- The spin vector orientation angle α, relative to the particle direction, can be most easily recovered after the particle has completed a single turn.

11 Global evolution of 2D spin vector **s** in E+M field

Jackson (11.170) gives the evolution equation (in MKS units) for a 2D unit spin vecctor **s**

$$\mathbf{s} = \cos\psi\hat{\mathbf{z}} - \sin\psi\hat{\mathbf{x}}$$
$$\frac{d\mathbf{s}}{dt} = (-\sin\psi\hat{\mathbf{z}} - \cos\psi\hat{\mathbf{x}})\,\frac{d\psi}{dt}, \tag{13}$$

where $d\mathbf{s}/dt$ has been calculated assuming uniform speed circular motion, always orthogonal to both $\mathbf{E} = -E\hat{\mathbf{x}}$ and $\mathbf{B} = B\mathbf{y}$. Using $\hat{\mathbf{z}} \times (-\hat{\mathbf{x}}) = -\hat{\mathbf{y}}$, the result is

$$\frac{d\mathbf{s}}{dt} = \frac{1}{mc/e}\,\mathbf{s} \times \hat{\mathbf{y}}\left(\left(G + \frac{1}{\gamma}\right)cB + \left(G + \frac{\gamma}{\gamma+1}\right)\beta E\right). \tag{14}$$

Substituting from (13) into (14), the same equation is obtained for each of the two non-vanishing vector components;

$$\frac{d\psi}{dt} = \frac{1}{mc/e}\left(\left(G + \frac{1}{\gamma}\right)cB + \left(G + \frac{1}{\gamma+1}\right)\beta E\right). \tag{15}$$

To caclulate tunes, in order to obtain $d\psi/d\theta$, we need $d\theta/dt$, to change independent variable $t \rightarrow \theta$, where $0 \leq \theta \leq 2\pi$ is circumferential angular coordinate around the ring. For our circular motion the factor needed is

$$\frac{dt}{d\theta} = \frac{p/e}{\beta cB + E} = \frac{\gamma\beta mc/e}{\beta cB + E}, \tag{16}$$

which (interestingly) is independent of radius r_0. We then obtain

$$\frac{d\psi}{d\theta} = \frac{1}{\beta cB + E}\left(\left(G\gamma + 1\right)\beta cB + \left(G\gamma + \frac{\gamma}{\gamma + 1}\right)\beta^2 E\right). \tag{17}$$

By setting either $E = 0$ or $B = 0$ one can obtain "tunes" $\tilde{Q}_s^E$ or $\tilde{Q}_s^M$, to check against results obtained previously for purely magnetic or purely electric bending;

$$\tilde{Q}_s^M = G\gamma + 1; \quad \tilde{Q}_s^E = G\gamma - \frac{G+1}{\gamma} + 1, \tag{18}$$

The offsets by 1 have come about because these tunes are referenced to ψ rather than to α (where the term "spin tune" is conventionally defined).

13 Returning to Eq.(17) and the superimposed electric and magnetic case, after using the pure bend results just mentioned and further simplification, we obtain

$$\begin{aligned}\tilde{Q}_s^{EM} &= \left(G\gamma \frac{\beta cB}{\beta cB + E} + \left(G\gamma + \frac{G+1}{\gamma} \right) \frac{E}{\beta cB + E} + 1 \right) \\ &= \left(G\gamma\, \eta_M + \left(G\gamma + \frac{G+1}{\gamma} \right) \eta_E + 1 \right), \end{aligned} \tag{19}$$

where η_E and $\eta_M = 1 - \eta_E$ are the bending fractions defined previously. Finally, subtracting the "1", we obtain a result used in a previous lecture—the (conventionally-defined) spin tune Q_s^{EM} with superimposed electric and magnetic bending.

$$Q_s^{EM} = Q_s^M \eta_M + Q_s^E \eta_E. \tag{20}$$

It is my suspicion that, starting with the 2009 BNL deuterium EDM proposal[4], deuterium EDM proposals, while "cancelling" electric and magnetic torques, have not correctly handled the spin tune in rings with superimposed electric and magnetic bending.

14 Solving the BMT equation (in pure electric field)

Substituting for the spin vector produces

$$\frac{d}{dt}\left(s_{\parallel}\cos\alpha\right) = -\frac{e}{m_p c}\left(s_{\parallel}\sin\alpha\, E\right)\left(\frac{g}{2}\beta - \frac{1}{\beta}\right). \tag{21}$$

- With the orbit confined to a plane, any precession occurs about the normal to the plane, conserving s_y.
- Since the magnitude of $\mathbf{s}$ is conserved it follows that the magnitude $s_{\parallel}$ is also conserved.
- This allows $s_{\parallel}$ to be treated as constant in this equation, which reduces to

$$\frac{d\alpha}{dt} = \frac{eE}{m_p c}\left(\frac{g}{2}\beta - \frac{1}{\beta}\right). \tag{22}$$

- Meanwhile the velocity vector itself has precessed by angle ϑ. The precession rate of ϑ is governed by the equation for circular motion, $pv/r = eE$;

$$\frac{d\vartheta}{dt} = \frac{v}{r} = \frac{eE}{p}, \tag{23}$$

- The independent variable can be switched, $t \rightarrow \vartheta \rightarrow \theta$ by dividing these two equations, and noting that, though ϑ and θ are not the same instantaneously, over long times they advance at the same average rate;

$$\frac{d\alpha}{d\theta} = \frac{pc}{m_p c^2}\left(\frac{g}{2}\beta - \frac{1}{\beta}\right) = \gamma\left(\frac{g}{2} - 1\right) - \frac{g}{2}\frac{1}{\gamma} \tag{24}$$

- Note that β has been eliminated in favor of γ.

- It is also conventional to express spin evolution in terms of "anomalous magnetic moment" $G = g/2 - 1$, yielding

$$\frac{d\alpha}{d\theta} = G\gamma - \frac{G+1}{\gamma}. \tag{25}$$

- The first term, $G\gamma$ can be recognized as the spin tune Q_s^M in a magnetic ring. The second term corrects for replacing magnetic bending by electric bending.
- Integrating over θ, the bend frame precession advance is the sum of two definite integrals

$$\Delta\alpha(\theta) = GI_\gamma(\theta) - \frac{G+1}{2} I_{\gamma i}(\theta), \tag{26}$$

where

$$I_\gamma(\theta) = \int_0^\theta \gamma(\theta')d\theta', \quad \text{and} \quad I_{\gamma i}(\theta) = \int_0^\theta \frac{d\theta'}{\gamma(\theta')}. \tag{27}$$

To account for fringe fields two more terms, $\Delta\alpha^{\mathrm{FF,in}}$ and $\Delta\alpha^{\mathrm{FF,out}}$, also need to be included eventually.

17 Re-derivation to include horizontal betatron displacement

- We need to account for the transverse position oscillations accompanying potential energy variation.
- For simplicity assume the lattice is uniform—no drift regions.
- The spin precession angle α, relative to the proton direction, evolves as

$$\frac{d\alpha}{dt} = \frac{eE(x)}{m_p c}\left(\frac{g\beta(x)}{2} - \frac{1}{\beta(x)}\right). \tag{28}$$

- The variables β, γ, and E in this equation are now being allowed to depend on x.
- Definition of angular momentum and its conservation yield

$$\frac{d\theta}{dt} = \frac{L}{\gamma m_p r^2}; \tag{29}$$

- This is valid in bend regions, but would not be in drift regions, where r becomes ambiguous.

- In this equation the angular momentum L is a constant of the motion (because the force is radial) but γ and $r = r_0 + x$ depend on x.
- Combining the two previous equations,

$$\frac{d\alpha}{d\theta} = \frac{E(x)(r_0 + x)^2}{\beta(x)Lc/e}\left(\left(\frac{g}{2} - 1\right)\gamma(x) - \frac{g/2}{\gamma(x)}\right), \qquad (30)$$

- To find the evolution of α over long times, for an individual particle, we need to average this equation.
- What makes this averaging difficult is the fact that the final factor has, intentionally, been arranged to cancel for the central, design particle.
- The initial factor, though not constant, varies over a quite small range.
- A promising approximation scheme for this factor is to neglect the (small) rapidly oscillating betatron contibution to x coming from the betatron oscillation

- We retain only the off-momentum part $x = D_x \Delta\gamma^O$ associated with the slowly varying synchrotron oscillation;

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle (\gamma^O) = \frac{eE(D_x\Delta\gamma^O)(r_0 + D_x\Delta\gamma^O)^2}{Lc\beta(D_x\Delta\gamma^O)} \left\langle \left(\frac{g}{2} - 1\right)\gamma(x) - \frac{g/2}{\gamma(x)} \right\rangle. \tag{31}$$

- No superscript "I" is needed on $\gamma(x)$ in the final factor since only "inside" motion is to be under discussion.
- If the average of $\langle\gamma\rangle$ were the inverse of $\langle 1/\gamma\rangle$ the averaging over horizontal betatron oscillation would be easy. But this is not true.
- However *the above factorization has allowed the averaging over "outside variable" γ^O to be deferred.*

20 Virial theorem decoherence calculation

- The virial theorem can be used to perform 3D averages over multiparticle systems subject to central forces.
- Also, though our electric field is centrally directed within any single deflecting element, because of drift regions in the lattice, the centers of the various deflection elements do not coincide.
- We can therefore calculate only the spin decoherence applicable to passage through the bend regions, which is where the overwhelmingly dominant part of the momentum evolution occurs.
- The independent variables θ and t are very nearly, but not exactly proportional to each other instantaneously, so averages with respect to one or the other are not necessarily identically instantaneously.
- However, with bunched beams over long times, θ and t are strictly proportional (on the average) and the two forms of averaging have to be essentially equivalent.
- Because there are so many variants of "the virial theorem" it is easier to derive it from scratch than to copy it from one of many possible references.

21 The "virial" G is defined, in terms of radius vector $\mathbf{r}$ and momentum $\mathbf{p}$, by

$$G = \mathbf{r} \cdot \mathbf{p} \tag{32}$$

Our electric field is

$$\mathbf{E} = -E_0 \left(\frac{r_0}{r} \right)^{1+m} \hat{\mathbf{r}}, \tag{33}$$

and Newton's law gives

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E}. \tag{34}$$

In a bending element the time rate of change of $\mathbf{G}$ is given by

$$\begin{aligned} \left. \frac{dG}{dt} \right|_{\text{bend}} &= \dot{\mathbf{r}} \cdot \mathbf{p} + \mathbf{r} \cdot \dot{\mathbf{p}} \\ &= m_p \gamma v^2 - eE_0 \, \frac{r_0^{1+m}}{r^m} \\ &= m_p c^2 \gamma - m_p c^2 \, \frac{1}{\gamma} - eE_0 r_0 \, \frac{r_0^m}{r^m}. \end{aligned}$$

Averaging over time, presuming bounded motion, and therefore requiring $\langle dG/dt \rangle$ to vanish, one obtains

$$\left\langle \frac{1}{\gamma} \right\rangle = \langle \gamma \rangle - \frac{E_0 r_0}{m_p c^2 / e} \left\langle \frac{r_0^m}{r^m} \right\rangle. \tag{35}$$

This provides the needed relation between $\langle \gamma \rangle$ and $\langle 1/\gamma \rangle$.

Applying this result to perform the (time)-average yields

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle = \frac{eE(D_x\Delta\gamma^O)(r_0 + D_x\Delta\gamma^O)^2}{Lc\beta(D_x\Delta\gamma^O)} \left(-\langle\gamma\rangle + \frac{g}{2}\,\frac{E_0 r_0}{m_p c^2/e} \left\langle \frac{r_0^m}{r^m} \right\rangle \right). \quad (36)$$

For specializing this result to frozen spin $\gamma = \gamma_0$ operation, the following formulas, can be employed:

$$\begin{aligned} \gamma(x) &\equiv \gamma_0 + \Delta\gamma, \\ \frac{E_0 r_0}{m_p c^2/e} &= \gamma_0 - \frac{1}{\gamma_0}, \\ \frac{r_0^m}{r^m} &\approx 1 - m\,\frac{x}{r_0}, \\ \gamma_0 &= \frac{g}{2}\left(\gamma_0 - \frac{1}{\gamma_0}\right). \end{aligned}$$

- These formulas assume the beam centroid energy and the storage ring lattice are exactly "magic". If not true the average spin orientation would change systematically. What is being calculated is the spin orientation spreading.
- For perfectly sinusoidal synchrotron oscilations, the initial factor can be replaced by its average value. This yields

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle \approx -\frac{E_0 r_0^2}{\beta_0 Lc/e} \left(\langle\Delta\gamma^I\rangle + \frac{g}{2}\,\frac{m}{r_0} \left(\gamma_0 - \frac{1}{\gamma_0}\right) \langle x\rangle \right). \quad (37)$$

(The superscript "I" has been restored as a reminder that $\Delta\gamma^I$ is evaluated within bend elements, as contrasted to within drift sections.) The numerical value of the leading factor is about 1.

- Copying the final equation from the previous slide, evaluating the leading factor on the design orbit, and dropping the negative sign, the decoherence rate is

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle = \langle \Delta\gamma' \rangle + \frac{g}{2}\, m \left(\gamma_0 - \frac{1}{\gamma_0} \right) \frac{\langle x \rangle}{r_0}. \tag{38}$$

- Typical values for the relevant quantities are

$$m = \pm 0.02, \tag{39}$$

$$\frac{x}{r_0} = \frac{0.01}{40} \approx 3 \times 10^{-4} \tag{40}$$

$$\Delta\gamma' = 3 \times 10^{-6} \tag{41}$$

- Small as they are, to linear approximation each of these averages to zero. To following order

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle \sim (3 \times 10^{-6})^2 + \frac{g}{2}\, 4 \times 10^{-4} \left(\gamma_0 - \frac{1}{\gamma_0} \right) (3 \times 10^{-4})^2. \tag{42}$$

- Decoherence in bend fringe fields is likely to be greater than this, but it also cancels if care is taken to assure linear synchrotorn oscillations.

- We have shown, therefore, for "weak-weaker" focusing, that decoherence in the bend regions can be neglected even in the presence of horizontal betatron oscillations,
- We have previously argued that decoherence associated with vertical betatron oscillation can also be neglected.
- Furthermore, very long spin coherence times have been demonstrated for deuterons in the COSY storage ring in Juelich, Germany, though only after quite delicate adjustment of nonlinear elements in the ring. Nevertheless COSY is a strong focusing ring for which spin decoherence can be expected to be far greater than in our weak focusing WW-AG-CF lattice.
- Conclusion: if beam bunches can for survive for many hours their polarization states can probably survive as well.

25 Spin evolution in idealized lattice

Except for idealized specializations, this section reproduces Sections II, III, and IV of Chapter'2 of S.Y. Lee's book "Spin Dynamics and Snakes in Synchrotrons", almost line by line. My purpose is to demonstrate the spectacular simplification that results when the design orbit can be assumed to remain in a single plane. A tiny, but confusing, further alteration is to use y (rather than Lee's z) as vertical coordinate. The coordinates (x, s, y), in (horizontal, longitudinal, vertical) order, (which for Lee is (x, s, z)) will be retained, in spite of the fact that more conventional in the storage ring world would be (horizontal, vertical, longitudinal). This may seem very confusing (partly because the longitudinal Frenet coordinate is to be s) but avoids mis-interpreting z as longitudinal, and it means that, to convert any Lee formula to a formula in this appendix, one only needs to make the replacement $z \rightarrow y$ (plus simplifying assumptions). What motivates this coordinate order for spin evolution is that it is useful for the first two coordinates (x, s) to be 2D coordinates in the (horizontal) ring design plane, with y being vertical.

- For simplicty in copying from Lee, **the discussion will be limited to ordinary (magnetic) rings.** The fundamental BMT spin evolution formula is (SYL-2.40)

$$\frac{d\mathbf{S}}{d\theta} = \mathbf{S} \times \mathbf{F}, \tag{43}$$

- θ is bend angle with $d\theta = ds/\rho$ in a dipole, ρ is radius of curvature, and the applied torque (multiplied by an appropriate factor) is (SYL-2.41)

$$\mathbf{F} = F_x\hat{\mathbf{x}} + F_s\hat{\mathbf{s}} + F_y\hat{\mathbf{y}}. \tag{44}$$

- In our idealized ring the design orbit lies in the (x, s) plane and $(y = 0, y' = 0, y'' = 0)$, where derivatives with respect to s are indicated by primes.
- Using these values, and copying from (SYL-2.42), the components of $\mathbf{F}$ are

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_s \\ F_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(1+G\gamma) + (1+G\gamma)\rho x'' \end{pmatrix}. \tag{45}$$

- S.Y. justifies dropping the x'' factor from F_y by stating that its average (presumably over betatron oscillation) is zero. The validity of this simplication will be returned to shortly.

27 Expressing **S** in terms of its components (SYL-2.43),

$$\mathbf{S} = S_x\hat{\mathbf{x}} + S_s\hat{\mathbf{s}} + S_y\hat{\mathbf{y}}, \tag{46}$$

and using $d\hat{\mathbf{x}}/d\theta = \hat{\mathbf{s}}$ and $d\hat{\mathbf{s}}/d\theta = -\hat{\mathbf{x}}$ we get (SYL-2.44)

$$\frac{d}{d\theta}\begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix} = \begin{pmatrix} (1+F_y)S_s \\ -(1+F_y)S_x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & (1+F_y) & 0 \\ -(1+F_y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix}. \tag{47}$$

This amounts to being the BMT equation, in S.Y. Lee notation, under our 2D assumptions. Continuing to follow Lee, we define (complex) spin components

$$S_\pm = S_x \pm iS_s, \quad \text{with inverses} \quad S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i}, \tag{48}$$

as well as field components $F_\pm = F_x \pm iF_s$ (that will not actually be needed under our special assumptions) to obtain (averaged) equations of motion,

$$\frac{d}{d\theta}\begin{pmatrix} S_\pm \\ S_y \end{pmatrix} = \begin{pmatrix} \pm iG\gamma S_\pm \\ 0 \end{pmatrix}. \tag{49}$$

Solving these differential equations, the spin evolution is given by (SYL-2.45);

$$S_{\pm} = e^{\pm iG\gamma\theta} S_{\pm 0}, \quad S_y = \text{constant}. \tag{50}$$

- One sees that $S_{\pm}$ are eigenfunctions of the propagation, with eigenvalues $e^{\pm iG\gamma\theta}$.
- Introduction of $S_{\pm}$ has further decoupled the spin motion.
- For propagation around the full ring, one sets $\theta = 2\pi$ and obtains $e^{\pm 2\pi iG\gamma}$ as the eigenvalues. It is because the eigenvalues are complex, while the S_i components are real, that the expansions of eigenfunctions as superpositions of S_i components has required a complex coefficient.
- As a self-consistency check, the spin tune of an "ideal" (magnetic) lattice has been shown again to be

$$Q_s = G\gamma. \tag{51}$$

- A curious feature of this relation is its lack of dependence on radius of curvature ρ.
- Ordinarily the radius of curvature ρ has roughly the same value in every bend element.
- But one could design a ring where ρ had different values in different sectors. This would leave relation $Q_s = G\gamma$ unaffected. To say, therefore, that $Q_s = G\gamma$ is a property of an "ideal' lattice, has significantly expanded the meaning of "ideal".
- A (horizontally) misaligned quadrupole steers the central orbit, but leaves the central closed orbit in the same (horizontal) design plane. Why don't we just pretend that the misalignment is part of the "design", in which case we still have $Q_s = G\gamma$.

- It seems to me, therefore, that $G\gamma$ is a global invariant, unaffected by lattice errors, to the extent they leave the central closed orbit in a single plane. (Operationally one can achieve this condition to high accuracy.)
- Small vertical betatron motion would not alter this conclusion to lowest order. Non-commutation of rotations could violate the averaging to zero, but only proportional to the product of already extremely small amplitudes.
- All this is consisrent with the earlier 2D virial theorem demonstration that spin decoherence can be neglected in our ultraweak focusing WW-AG-CF lattice. Once a beam bunch has been "captured" its total spin precesses almost as if the bunch were a single particle.

31 Spinor representation of spin evolution

Continuing to specialize formulas from S.Y. Lee, our spin evolution equation has decoupled to two equations (SYL-2.46)

$$\frac{d\mathbf{S}_{\|}}{d\theta} = G\gamma\,\hat{\mathbf{y}} \times \Big(S_x\hat{\mathbf{x}} + S_s\hat{\mathbf{s}} + S_y\hat{\mathbf{y}}\Big) = G\gamma\Big(S_x\hat{\mathbf{s}} - S_s\hat{\mathbf{x}}\Big),$$
$$\frac{dS_y}{d\theta} = 0.$$

Instead of the 3-component real vectors used so far, one can represent the spin by a 2-complex component general spinor (SYL-2.51),

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix}, \tag{52}$$

Introducing the Pauli matrices, as the three components of a "vector" $\boldsymbol{\sigma}$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_s = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{53}$$

the 3D spin components can be expressed in terms of ψ as (SYL-2.47)

$$S_i = \Psi^\dagger \sigma_i \Psi, \quad \text{for} \quad i = x, s, y. \tag{54}$$

- This permits the S_1 components to be expressed in terms of u and d, (SYL-2.52);

$$S_1 = u^* d + u d^*, \quad S_2 = -i(u^* d - u d^*), \quad S_3 = |u|^2 - |d|^2. \tag{55}$$

- Even though u and d are complex in general, the S_i components are always real. With this notation, the spin evolution equation becomes (SYL-2.48)

$$\frac{d\boldsymbol{\Psi}}{d\theta} = -\frac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\Psi}) = -\frac{i}{2}\mathbf{H}\boldsymbol{\Psi}, \quad \text{where} \quad \mathbf{H} = \begin{pmatrix} G\gamma & 0 \\ 0 & -G\gamma \end{pmatrix} = G\gamma\sigma_y. \tag{56}$$

- Notice that Lee's perturbation function (SYL-2.49), $\xi(\theta) = F_x(\theta) - iF_s(\theta)$, which would have appeared in the off-diagonal elements of $\mathbf{H}$, actually vanishes identically (as a consequence of our pure-planar design orbit assumption).
- For a lattice with arbitrary errors the full formalism continues to be valid, in which case, of course, the off-diagonal elements of $\mathbf{H}$ need to be correctly included.

33 Spinor transfer matrix

- For initial polarization state $\boldsymbol{\Psi}(\theta_1)$, solving this differential equation produces a later polarization state $\boldsymbol{\Psi}(\theta_2)$ given by (SYL-2.54)

$$\boldsymbol{\Psi}(\theta_2) = e^{-\frac{i}{2}\,\mathbf{H}\,(\theta_2-\theta_1)}\,\boldsymbol{\Psi}(\theta_1) \equiv t(\theta_2,\theta_1)\,\boldsymbol{\Psi}(\theta_1). \tag{57}$$

  which defines $t(\theta_2,\theta_1)$ as a "transfer matrix" in the spinor formalism.

- With $\mathbf{H}$ diagonal, the spin evolves as

$$t(\theta_2,\theta_1) = e^{-\frac{i}{2}\,G\gamma(\theta_2-\theta_1)\,\sigma_y}. \tag{58}$$

- This evolution formula resembles the earlier $S_\pm$ eigenfunction evolution formula closely, but with the important difference that the matrix σ_y appears here in the exponent. This does not seriously complicate algebraic manipulations for which this result is to be applied.
- All this "looks like" quantum mechanics, but it has just been algebraic manipulation.

34 One turn map (OTM)

- Also useful is the one turn transfer map (OTM), $t(\theta)$ from arbitrary initial angle θ, once around the ring and back to the same location.
- It satisfies (SYL-2.58)

$$\boldsymbol{\Psi}(\theta + 2\pi) = \Pi_{j=1}^{N}\, t(\theta_{j+1}, \theta_j)\, \boldsymbol{\Psi}(\theta), = t(\theta)\, \boldsymbol{\Psi}(\theta), \qquad (59)$$

  which concatenates the transfer maps over all N ring sectors starting from θ and returning to the same position.

35 Storage ring as "Penning-Like Trap"—motivation

- The possibility of storing a large number, such as 10^{10}, of identically polarized particles makes a storage ring an attractive charged particle "trap".
- But, compared to a table top trap, a storage ring is a quite complicated assemblage of many carefully, but imperfectly, aligned components, powered from not quite identical sources.
- Fortunately, particle magnetic dipole moments (MDM) have been measured to exquisitely high precision. For our purposes MDM's can be treated as exactly known.
- High enough beam polarization, and long enough spin coherence time SCT, have made it possible to "freeze" the spins.

- In this frozen state, the importance of some inevitable machine imperfections, that might otherwise be expected to dominate the errors, is greatly reduced. Examples are beam energy spread and ring element positioning and alignment uncertainties.
- (With the benefit of RF-imposed synchrotron oscillation stability) the average beam energy is fixed with the same exquisitely high accuracy with which the MDM is known.
- The polarization vector serves as the needle of a perfect speedometer. With the RF frequency also known to exquisite accuracy, the revolution period is similarly well known.
- Then, irrespective of element locations and powering errors, the central orbit circumference is, if not perfectly known, is at least known to be constant in time (except for knowable and controllable element changes).

37 Small deviations from magic condition

- To encompass all of these considerations the storage ring can be referred to as a "polarized beam trap".
- This strategem allows some sources of error to less terrifying, but without eliminating them altogether. Of course one will build the EDM storage ring as accurately as possible.
- The lack of concern about element absolute positioning must not to be confused as lack of concern for BPM, orbit positioning precision, even assuming the ring has been tuned to be a perfect trap.
- Reduction of systematic EDM measurement error will depend critically on precision beam positioning control.
- Essential to this way of conceptualizing the experiment is that the constancy in time of the apparatus parameters is what is important, rather than the degree to which their absolute values have ideal values.

38 Small deviations from magic condition

- Suppose the beam magnetization phase has been locked by external feedback, for example in an electron ring. One can then take advantage of the precisely-known electron magnetic moment μ_e and anomalous moment G_e. The spin tune Q_s^E relates to precession around the vertical axis. In an all-electric ring Q_s^E is given by

$$Q_s^E = G_e\gamma - \frac{G_e + 1}{\gamma},$$

For frozen spin electrons at the "magic" value, $Q_s^E = 0$, $\gamma = \gamma_m$, where

$$\gamma_m = \sqrt{\frac{G_e + 1}{G_e}} = 29.38243573. \tag{60}$$

Solving for γ, (and requiring $\gamma > 0$),

$$\gamma = \frac{Q_s^E + \sqrt{{Q_s^E}^2 + 4G_e(G_e + 1)}}{2G_e}, \tag{61}$$

we then obtain

$$\gamma = 29.38243573 + 431.16379, \frac{\Delta f_y}{f_0} + 3163.5 \left(\frac{\Delta f_y}{f_0}\right)^2 + \cdots .$$

Here Q_s^E has been re-expressed in terms of the frequency deviation from magic, $\Delta f_y = f_y - f_m$, of the polarization around a vertical axis. This formula is intended for use only near γ_m, with the ratio $\Delta f_y / f_0$ being a tiny number, less than 10^{-5} for example.


J.D. Jackson, *Classical Electrodynamics,* 3rd edition, John Wiley, 1999


S.Y. Lee, *Spin Dynamics and Snakes in Synchrotrons* World Scientific, 1997


Storage Ring EDM Collaboration, *A Proposal to Measure the Proton Electric Dipole Moment with* 10^{-29} *e-cm Sensitivity,* October, 2011


V. Anastassopoulus, et al. *Search for a permanent electric dipole moment of the deuteron*, AGS proposal, 2008