Outline

- Electric Dipole Moments (EDM)
  - Observable: Polarisation
    - Optimal Observables, Event Weighting, Maximum Likelihood Method
Electric Dipole Moments (EDM)

- permanent separation of positive and negative charge
- fundamental property of particles (like magnetic moment, mass, charge)
- existence of EDM only possible via violation of time reversal $\mathcal{T}$ and parity $\mathcal{P}$ symmetry
- close connection to “matter-antimatter” asymmetry
- axion field leads to oscillating EDM
$N$ BARYONS

$(S = 0, I = 1/2)$

$p, N^+ = uud; \quad n, N^0 = udd$

$I(J^P) = \frac{1}{2}(\frac{1}{2}^+)$

Mass $m = 1.00727646688 \pm 0.0000000009$ u

Mass $m = 938.272081 \pm 0.000006$ MeV [a]

$|m_p - m_p|/m_p < 7 \times 10^{-10}$, CL = 90% [b]

$q_e^2/m_p^2 / (q_e^2/m_p^2) = 1.0000000000 \pm 0.0000000007$

$q_p + q_p / |e| < 7 \times 10^{-10}$, CL = 90% [b]

$q_p + q_e / |e| < 1 \times 10^{-21}$ [c]

Magnetic moment $\mu = 2.7928473446 \pm 0.0000000008 \mu_N$

$(\mu_p - \mu_p) / \mu_p = (0.3 \pm 0.8) \times 10^{-6}$

Electric dipole moment $d < 0.021 \times 10^{-23}$ e cm

Electric polarizability $\alpha = (11.2 \pm 0.4) \times 10^{-4}$ fm$^3$

Magnetic polarizability $\beta = (2.5 \pm 0.4) \times 10^{-4}$ fm$^3$ $(S = 1.2)$

Charge radius, $\mu_p$ Lamb shift = 0.84087 ± 0.00039 fm [d]

Charge radius, $e_p$ CODATA value = 0.8751 ± 0.0061 fm [d]

Magnetic radius = 0.851 ± 0.026 fm [e]

Mean life $\tau > 2.1 \times 10^{29}$ years, CL = 90% [f] ($p \rightarrow$ invisible mode)

Mean life $\tau > 10^{31}$ to $10^{33}$ years [f] (mode dependent)
EDM: Current Upper Limits

![Chart showing EDM limits for various particles](image)

**Experimental Limits**

- QCD $\theta_{\text{QCD}} = 0$
- SM $\theta_{\text{SM}} < 1$
- CP $\varphi_{\text{CP}} < \pi$
- SUSY $\frac{\alpha}{\pi} < \varphi_{\text{CP}} < 1$

**Storage Rings:** EDMs of **charged** hadrons: $p, d, ^3\text{He}$
Experimental Method: Generic Idea

The build-up of vertical polarization $s_\perp \propto d$, if $\vec{s}_{\text{horz}} || \vec{p}$ (frozen spin)

\[
\frac{d\vec{s}}{dt} \propto d(\vec{E} + \vec{v} \times \vec{B}) \times \vec{s} = \vec{\Omega}_{\text{EDM}}
\]
Experimental Method: Generic Idea

\[ \frac{d\vec{s}}{dt} \propto d(\vec{E} + \vec{v} \times \vec{B}) \times \vec{s} = \vec{\Omega}_{EDM} \]

In general:

\[ \frac{d\vec{s}}{dt} = (\vec{\Omega}_{MDM} + \vec{\Omega}_{EDM}) \times \vec{s} \]

build-up of vertical polarization \( s_{\perp} \propto d \), if \( \vec{s}_{\text{horz}} \parallel \vec{\rho} \) (frozen spin)
\[
\frac{d\vec{s}}{dt} = \vec{\Omega} \times \vec{s} = \frac{-q}{m} \left[ G\vec{B} + \left( G - \frac{1}{\gamma^2 - 1} \right) \vec{v} \times \vec{E} + \frac{\eta}{2} (\vec{E} + \vec{v} \times \vec{B}) \right] \times \vec{s}
\]

Electric dipole moment (EDM): \( \vec{d} = \eta \frac{q\hbar}{2mc} \vec{s} \),

Magnetic dipole moment (MDM): \( \vec{\mu} = 2(G + 1) \frac{q\hbar}{2m} \vec{s} \)

Note: \( \eta = 2 \cdot 10^{-15} \) for \( d = 10^{-29} \) ecm, \( G \approx 1.79 \) for protons
Spin Precession: Thomas-BMT Equation

\[ \frac{d\vec{s}}{dt} = \vec{\Omega} \times \vec{s} = \frac{-q}{m} \left[ GB + \left( G - \frac{1}{\gamma^2 - 1} \right) \vec{v} \times \vec{E} + \frac{\eta}{2} (\vec{E} + \vec{v} \times \vec{B}) \right] \times \vec{s} \]

\[ \vec{\Omega}_{MDM} = 0, \text{ frozen spin } \]

\[ \vec{\Omega}_{EDM} = \vec{\Omega}_{EDM} \]

achievable with pure electric field if \( G = \frac{1}{\gamma^2 - 1} \), works only for \( G > 0 \), e.g. proton

or with special combination of \( E, B \) fields and \( \gamma \), i.e. momentum
Momentum and ring radius for proton in frozen spin condition

\[ G = 1.7928474 \]
Momentum and ring radius for **deuteron** in frozen spin condition

\[ G = -0.1425617689 \]

\[ \begin{align*}
E/(\text{MV/m}) & \quad 0.0 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\
B/T & \quad 0.00 & 0.05 & 0.10 & 0.25 & 0.50 & 1.00 \\
p/(\text{GeV}/c) & \quad 0.05 & 0.10 & 0.25 & 0.50 & 1.00 & 2.00 & 3.00 & 5.00 & 10.00 \\
\text{radius/m} & \quad 1.00 & 5.00 & 10.00 & 20.00 & 30.00 & 40.00 & 50.00 & 75.00 & 100.00 & 1000.00
\end{align*} \]
Momentum and ring radius for electron in frozen spin condition

\[ G = 0.001159652 \]

\[ B/T \]

\[ p/(GeV/c) \]

\[ \text{radius/m} \]

\[ E/(MV/m) \]
# Different Options

<table>
<thead>
<tr>
<th>Option</th>
<th>Advantage</th>
<th>Disadvantage</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.) pure electric ring</td>
<td>no $\vec{B}$ field needed, $\bigcirc$, $\bigcirc$ beams simultaneously</td>
<td>works only for particles with $G &gt; 0$ (e.g. $e, p$)</td>
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<tr>
<td>2.) combined ring</td>
<td>works for $e, p, d, ^3$He, smaller ring radius</td>
<td>both $\vec{E}$ and $\vec{B}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$B$ field reversal for $\bigcirc$, $\bigcirc$ required</td>
</tr>
<tr>
<td>1.) pure magnetic ring</td>
<td>existing (upgraded) COSY ring can be used, shorter time scale</td>
<td>lower sensitivity, precession due to $G$, i.e. no frozen spin</td>
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Observable is in all cases a **spin polarization**!

**Talk on EDM**
Polarization Measurements
Counting Rates, Cross Section, Polarization

\[ N(\vartheta, \varphi) = a(\vartheta, \varphi) \cdot L \cdot \sigma(\vartheta) \left( 1 + P \cdot A(\vartheta) \cdot \cos(\varphi) \right) \]

- number of observed events
- acceptance/efficiency
- luminosity
  \( L = \text{beam flux} \times \text{target density} \times \text{target length} \)
- unpolarized cross section
- beam polarisation
  \( P = \frac{n_{\uparrow} - n_{\downarrow}}{n_{\uparrow} + n_{\downarrow}} \)
- analysing power
  \( A = \frac{\sigma_{L} \uparrow - \sigma_{R} \uparrow}{\sigma_{L} \uparrow + \sigma_{R} \uparrow} \)
polarisation: \[ P = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow} = \frac{3 - 2}{3 + 2} = 0.2, \]

analyzing power \[ A = \frac{\sigma^\uparrow_L - \sigma^\uparrow_R}{\sigma^\uparrow_L + \sigma^\uparrow_R}. \]

Note: \[ \sigma^\uparrow_L \equiv \sigma^\downarrow_R. \]

\[ N_L \propto (n^\uparrow \sigma^\uparrow_L + n^\downarrow \sigma^\downarrow_L) \]

\[ \Rightarrow N(\vartheta, \varphi) = \mathcal{L} a(\vartheta, \varphi) \sigma(\vartheta) \left(1 + P A(\vartheta) \cos(\varphi)\right), \quad \sigma = \frac{1}{2}(\sigma_L + \sigma_R) \]
Goal

Determine $P$ from counting rate $N(\vartheta, \varphi)$ and analysing power $A(\vartheta)$ with small uncertainty $\sigma_P$.

To simplify the discussion:

- Assume constant acceptance in $\varphi$: $\frac{\partial a(\vartheta, \varphi)}{\partial \varphi} = 0$
- Detector placed at one polar angle $\vartheta$

We are left with

$$N(\varphi) = \frac{1}{2\pi} N_0 (1 + PA \cos(\varphi)) \quad , \quad N_0 = a \mathcal{L} \sigma$$
Event Distribution

\[ \varphi = 0 \]

nb. of events/a.u.
Most easy way to get $P$

Just consider counts in the left part of the detector $\varphi \approx 0, \cos(\varphi) = 1$ and the right part $\varphi \approx \pi, \cos(\varphi) = -1$.

\[
\langle N_L \rangle = N_0 \frac{\Delta \varphi}{2\pi} (1 + AP)
\]
\[
\langle N_R \rangle = N_0 \frac{\Delta \varphi}{2\pi} (1 - AP)
\]

Consider a counting rate asymmetry

\[
\hat{P} = \frac{1}{A} \frac{N_L - N_R}{N_L + N_R}, \quad \hat{P}:\text{ estimator for } P.
\]

If $A$ is known, one can determine $P$.

Note:

$\langle N_{L,R} \rangle$: expectation value

$N_{L,R}$: actually measured number of events
What about the error?

Error propagation gives: \( \sigma_P = \frac{1}{A \sqrt{N}} \)

(assuming \( PA \ll 1 \), i.e. \( N_L \approx N_R =: N/2 \))

As in any counting experiment the statistical error scales with \( 1/\sqrt{N} \).

Counting only events in small region \( \Delta \varphi \) around \( \varphi = 0 \) and \( \pi \) results in small \( N = N_0 \frac{2 \Delta \varphi}{2\pi} \) and thus large error.

It’s more convenient to work with the Figure of Merit (FOM):

\[ \text{FOM}_P = \sigma_P^{-2} = N A^2 \]

How does error change if we include more events, i.e. making \( \Delta \varphi \) larger?
Enlarge $\varphi$ range
$$\hat{P} = \frac{1}{N_L - N_R} \frac{N_L - N_R}{N_L + N_R}$$

$$\sigma_P = \frac{1}{\sqrt{N}} \frac{1}{A\langle \cos(\varphi) \rangle} ,$$

number of events: $$N = \frac{4\varphi_{\text{max}}}{2\pi}$$

$$\varphi_{\text{max}} \uparrow \Rightarrow N \uparrow$$

$$\varphi_{\text{max}} \uparrow \Rightarrow \langle \cos(\varphi) \rangle \downarrow$$

$$\langle \cos(\varphi) \rangle = \frac{\int_{-\varphi_{\text{max}}}^{\varphi_{\text{max}}} \cos(\varphi) d\varphi}{2\varphi_{\text{max}}}$$

$$\text{FOM}_P = \sigma_P^{-2} = N (A\langle \cos(\varphi) \rangle)^2$$
Figure of Merit (FOM)

- strange behavior: Adding data beyond $\varphi_{\text{max}} > 67^\circ$ the FOM decreases
- Reason: adding data at larger $\varphi$ “dilutes” the sample
Can one do better? Yes! **Event Weighting**

Instead of just counting events, weight every event with a weight function $w(\varphi)$.

Estimator for $P$

$$\hat{P} = \frac{1}{A} \frac{\sum_{L,R} w_i}{\sum_{L,R} w_i \cos(\varphi_i)}$$

In principle weight $w$ arbitrary, two cases are of interest

- $w = 1$ (left), $w = -1$ (right): $\hat{P} = \frac{1}{A \langle \cos(\varphi) \rangle} \frac{N_L - N_R}{N_L + N_R}$ (counting rate asymmetry)

- $w = A \cos(\varphi)$: $\hat{P} = \frac{1}{A} \frac{\sum_{L,R} \cos(\varphi_i)}{\sum_{L,R} \cos^2(\varphi_i)}$ (optimal weight)$^1$

choice $w(\varphi) \equiv A \cos(\varphi)$ leads to smallest statistical error.

$^1$ In terms of highest FOM.
Every event weighted with $w = 1$

\[
\hat{P} = \frac{1}{A\langle \cos(\varphi) \rangle} \frac{N_L - N_R}{N_L + N_R}
\]
Every event weighted with \( w = A \cos(\varphi) \)
What about the error?

Error Propagation: $FOM_P = NA^2 \frac{\langle w \cos(\phi) \rangle^2}{\langle w^2 \rangle}$

- counting, $w = 1$
- $w = A\cos(\phi)$, MLH, binning

$FOM_P \quad NA^2 \langle \cos(\phi) \rangle^2 \quad NA^2 \langle \cos(\phi)^2 \rangle$

Gain in FOM: $\frac{\langle \cos(\phi)^2 \rangle}{\langle \cos(\phi) \rangle^2} \geq 1$

An event with a large $\cos(\phi)$ tells you more about $P$ than an event with lower $\cos(\phi)$. It should thus enter the analysis with more weight.
Connection to Maximum Likelihood Method

\[ N(\varphi) \propto (1 + A \cos(\varphi)P) = (1 + \beta(\varphi)P), \]
Here: \( \beta(\varphi) = A \cos(\varphi) \)

Log-likelihood function

\[ \ell = \sum_{i=1}^{N} \ln (1 + \beta(\varphi_i)P) \]
Connection to Maximum Likelihood Method

MLH estimator for $P$: Maximize $\ell \Rightarrow \frac{\partial \ell}{\partial P} = 0$

$$\Rightarrow \frac{\partial \ell}{\partial P} = \sum_i \frac{\beta(\varphi_i)}{1 + \beta(\varphi_i)P} = 0$$

for $\beta(\varphi_i)P \ll 1$:

$$\Rightarrow \sum_i \beta(\varphi_i)(1 - \beta(\varphi_i)P) = 0$$

$$\Rightarrow \hat{P} = \frac{\sum_i \beta(\varphi_i)}{\sum_i \beta^2(\varphi_i)} = \frac{1}{A} \frac{\sum_i \cos(\varphi_i)}{\sum_i \cos^2(\varphi_i)}$$

Estimator of maximum likelihood function coincides with estimator for optimal weight!

It is well known that MLH estimator reach largest FOM (Cramer-Rao-bound).
More general case

events follow distribution $N(\vec{x}) \propto (1 + \beta(\vec{x})P)$

For optimal event weight/MLH FOM is given by

$$FOM_P = N\langle \beta(\vec{x})^2 \rangle$$

Counting rates reach only

$$FOM_P = N\langle \beta(\vec{x}) \rangle^2$$

$$\langle \beta(\vec{x}) \rangle = \frac{\int_X \beta(\vec{x}) dx^n}{\int_X dx^n}, \quad X = \text{acc. events}$$

for example $\beta(\vec{x}) = \beta(\vartheta, \varphi) = A(\vartheta) \cos(\varphi)$
Polarizations can be extracted from azimuthal dependent event rates, knowing the analyzing power $A$

weighting the events with $\cos(\varphi)$ give the largest FOM

Gain with respect to just counting events is

$$\frac{\text{FOM}_{w=A\cos(\varphi)}}{\text{FOM}_{\text{cnt}}} = \frac{\langle \cos(\varphi)^2 \rangle}{\langle \cos(\varphi) \rangle^2}$$

Assumption made on acceptance, $PA \ll 1$, fixed $\vartheta$, ... were only made to simplify discussions


Spare
Polarization $P$, Analyzing Power $A$

\[ N_L = a_L \rho \ell (n^\uparrow \sigma^\uparrow, L + n^\downarrow \sigma^\downarrow, L) \]
\[ \varphi^{-\text{sym}} = a_L \rho \ell (n^\uparrow \sigma^\uparrow, L + n^\downarrow \sigma^\uparrow, R) \]
\[ N_R = a_R \rho \ell (n^\uparrow \sigma^\uparrow, R + n^\downarrow \sigma^\downarrow, R) \]
\[ \varphi^{-\text{sym}} = a_R \rho \ell (n^\uparrow \sigma^\uparrow, R + n^\downarrow \sigma^\uparrow, L) \]

\[ n^\uparrow (n^\downarrow) : \text{nb. of beam particles with spin up (down)} \]
\[ P = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow} : \text{Polarization} \]
\[ \sigma^\uparrow, R \equiv \sigma^\downarrow, L =: \sigma_R : \text{cross section for scattering process to} \]
\[ \text{the right (left) if spin is up (down)} \]
\[ \sigma^\downarrow, R \equiv \sigma^\uparrow, L =: \sigma_L : \]
\[ A = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L} : \text{analyzing power} \]
\[ \sigma_{\uparrow}, R = \sigma_{\downarrow}, L \]
With the definitions

\[ A = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L} \quad \text{and} \quad P = \frac{n^\uparrow - n^\downarrow}{n^\uparrow + n^\downarrow} \]

one can write

\[ N_R = a_R \rho \ell \sigma (1 + AP) \]
\[ N_L = a_L \rho \ell \sigma (1 - AP) \]

with \( \sigma = \frac{1}{2}(\sigma_R + \sigma_L) \)