

# Spin Evolution and Decoherence

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## 2 Outline

Spin evolution, approximately horizontal orbits

Spin evolution through bends

Spin decoherence

Spin evolution in idealized lattice

Storage ring as “Penning-Like Trap”

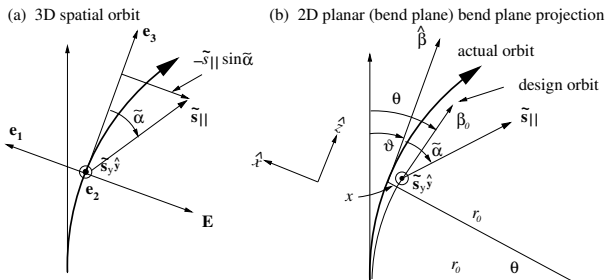
Pseudo-frozen spin magnetic ring parameters for e, p, d and He<sup>3</sup>

Magnetic ring spin tune alteration using Siberian snakes

Can pseudo-frozen spin satisfy  $Q_s = 0$  in magnetic ring? Yes.

Can snakes enable EDM measurement in magnetic ring? No.

### 3 Spin evolution, approximately horizontal orbits



- ▶ (a) Spatial, 3D orbit and spin evolution. The spin vector  $\mathbf{s}$  has precessed through angle  $\tilde{\alpha}$  away from its nominal direction along the proton's velocity.
- ▶ (b) Projection of figure (a) onto the 2D laboratory horizontal plane.  $x$  is the deviation of the (bold face) particle orbit from the (pale face) design orbit.
- ▶ If the bend plane coincides with the design bend plane (as is always approximately the case)  $\hat{\beta}_0$  and  $\hat{\mathbf{z}}$  are identical.
- ▶  $\theta$  is the reference particle global horizontal angle and  $\vartheta$  is the tracked particle global horizontal angle .
- ▶ Betatron oscillations cause  $\theta$  and  $\vartheta$  to differ (slightly) on a turn by turn basis but, on the average, they are the same

- ▶ Though the spin vector has three components, only two are independent;

$$\tilde{s}_{\parallel}^2 + \tilde{s}_{\perp}^2 = 1.$$

- ▶ The angle  $\tilde{\alpha}$  fixes the direction of  $\tilde{\mathbf{s}}_{\parallel}$  in the bend plane. Since we are assuming the bend plane and design plane are almost parallel, the tildes can pretty much be ignored.
- ▶ The bend frame spin coordinates are

$$\begin{pmatrix} \tilde{s}_x \\ \tilde{s}_y \\ \tilde{s}_z \end{pmatrix} = \begin{pmatrix} -\tilde{s}_{\parallel} \sin \tilde{\alpha} \\ \tilde{s}_{\perp} \\ \tilde{s}_{\parallel} \cos \tilde{\alpha} \end{pmatrix}.$$

- ▶ The spin precesses about the  $\tilde{y}$ -axis in the bend plane.
- ▶ These coordinates are ideal for evolving the spins through the (dominant) ring horizontal bending elements.

- 5 ▶ From particle tracking one knows the laboratory frame vectors  $\mathbf{r}$ ,  $\mathbf{p}$ , and angular momentum  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , and one also has the spin vector  $\mathbf{s}$ ;

$$\mathbf{r} = r_x \hat{\mathbf{x}} + r_y \hat{\mathbf{y}} + r_z \hat{\mathbf{z}},$$

$$\mathbf{p} = p_x \hat{\mathbf{x}} + p_y \hat{\mathbf{y}} + p_z \hat{\mathbf{z}},$$

$$\mathbf{L} = L_x \hat{\mathbf{x}} + L_y \hat{\mathbf{y}} + L_z \hat{\mathbf{z}},$$

$$\mathbf{s} = s_x \hat{\mathbf{x}} + s_y \hat{\mathbf{y}} + s_z \hat{\mathbf{z}}.$$

Any spin precession in the interior of a bend element occurs in the bend plane. To exploit this reduction from 3D to 2D it is necessary to obtain the spin components in an orthonormal frame having its “y” axis perpendicular to the plane and its “z” coordinate tangential to the orbit. In the (always excellent) paraxial approximation, this transformation is always close to identity.

- ▶ A right-handed basis triad has axis 3 parallel to  $\mathbf{p}$  and axis 2 parallel to  $-\mathbf{L}$ , where  $L$  is angular momentum and the negative sign is appropriate for clockwise orbits;

$$\mathbf{e}_3 = \frac{p_x}{p} \hat{\mathbf{x}} + \frac{p_y}{p} \hat{\mathbf{y}} + \frac{p_z}{p} \hat{\mathbf{z}},$$

$$\mathbf{e}_2 = \frac{\mathbf{r} \times \mathbf{p}}{-L},$$

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3.$$

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These equations can be re-expressed in terms of known coefficients

$$\mathbf{e}_1 = a_{11}\hat{\mathbf{x}} + a_{12}\hat{\mathbf{y}} + a_{13}\hat{\mathbf{z}}$$

$$\mathbf{e}_2 = a_{21}\hat{\mathbf{x}} + a_{22}\hat{\mathbf{y}} + a_{23}\hat{\mathbf{z}}$$

$$\mathbf{e}_3 = a_{31}\hat{\mathbf{x}} + a_{32}\hat{\mathbf{y}} + a_{33}\hat{\mathbf{z}}.$$

The vector  $\mathbf{s}$  can be expanded as

$$\begin{aligned}\mathbf{s} &= \tilde{s}_1\mathbf{e}_1 + \tilde{s}_2\mathbf{e}_2 + \tilde{s}_3\mathbf{e}_3 \\ &= \tilde{s}_1(a_{11}\hat{\mathbf{x}} + a_{12}\hat{\mathbf{y}} + a_{13}\hat{\mathbf{z}}) + \dots \\ &= (a_{11}\tilde{s}_1 + a_{21}\tilde{s}_2 + a_{31}\tilde{s}_3)\hat{\mathbf{x}} + \dots\end{aligned}$$

- ▶ The final relation can be expressed in matrix form as

$$\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} = \mathbf{R} \begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \end{pmatrix},$$

where  $\mathbf{R}$  is an orthogonal matrix,

$$\mathbf{R} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

(Aside: the magnitude  $|\det \mathbf{R}|$  of the determinant of  $\mathbf{R}$  is necessarily 1, but the actual value is  $\pm 1$ . This sign correlates with the clockwise/counterclockwise orbit ambiguity.)

Because  $\mathbf{R}$  is orthogonal,  $\mathbf{R}^{-1} = \mathbf{R}^T$  and Eq. (7) can be inverted to give

$$\begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}.$$

This yields the spin components in the bend frame. Their propagation through the bend is described next.

## 8 Spin evolution through bends

- ▶ For motion restricted to a single plane (which is implicit in the present discussion) the BMT equation can be solved exactly in closed form.
- ▶ In this frame any precession of the spin is purely around an axis normal to the plane. Because of ultraweak vertical focusing, (e.g. in the WW-AG-CF lattice) vertical betatron oscillations are negligible for the 2D evolution through electric bend elements. Any betatron oscillations actually present are treated as exactly horizontal.
- ▶ The initial spin vector is

$$\mathbf{s} = -s_{\parallel} \sin \alpha \hat{\mathbf{x}} + s_y \hat{\mathbf{y}} + s_{\parallel} \cos \alpha \hat{\mathbf{z}}.$$

Here  $s_y \hat{\mathbf{y}}$  is the out-of-plane component of  $\mathbf{s}$ ,  $s_{\parallel}$  is the magnitude of the in-plane projection of  $\mathbf{s}$ , and  $\alpha$  is the angle between the projection of  $\mathbf{s}$  onto the plane and the tangent vector to the orbit.



Jackson gives the rate of change in an electric field  $\mathbf{E}$ , of the longitudinal spin component as

$$\frac{d}{dt} (\hat{\beta} \cdot \mathbf{s}) = -\frac{e}{m_p c} (\mathbf{s}_{\perp, J} \cdot \mathbf{E}) \left( \frac{g\beta}{2} - \frac{1}{\beta} \right).$$

Substituting for the spin vector produces

$$\frac{d}{dt} (s_{\parallel} \cos \alpha) = -\frac{e}{m_p c} (s_{\parallel} \sin \alpha E) \left( \frac{g\beta}{2} - \frac{1}{\beta} \right).$$

With the orbit confined to a plane, any precession occurs about the normal to the plane, conserving  $s_y$ . Since the magnitude of  $\mathbf{s}$  is conserved it follows that the magnitude  $s_{\parallel}$  is also conserved. This allows  $s_{\parallel}$  to be treated as constant this equation, which reduces to

$$\frac{d\alpha}{dt} = \frac{eE}{m_p c} \left( \frac{g\beta}{2} - \frac{1}{\beta} \right).$$

Meanwhile the velocity vector itself has precessed by angle  $\vartheta$  relative to a direction fixed in the laboratory. The precession rate of  $\vartheta$  is governed by the equation

$$\frac{d\vartheta}{dt} = \frac{v}{r} = \frac{eE}{p},$$

The independent variable can be switched from  $t$  to  $\vartheta$  by dividing the previous two equations

$$\boxed{\frac{d\alpha}{d\vartheta} = \left(\frac{g}{2} - 1\right)\gamma - \frac{g/2}{\gamma}.}$$

In this step I have also surreptitiously made the replacement  $\vartheta \rightarrow \theta$ . Though these angles are not instantaneously the same, over long times they advance at the same average rate.

- 11 Integrating over  $\theta$ , the bend frame precession advance is the sum of two definite integrals

$$\Delta\alpha = \left(\frac{g}{2} - 1\right) l_\gamma - \frac{g}{2} l_{\gamma i},$$

where

$$l_\gamma = \int_0^\theta \gamma(\theta') d\theta', \quad \text{and} \quad l_{\gamma i} = \int_0^\theta \frac{d\theta'}{\gamma(\theta')}.$$

To account for fringe fields two more terms,  $\Delta\alpha^{\text{FF,in}}$  and  $\Delta\alpha^{\text{FF,out}}$ , need to be included eventually.

A quicker yet equivalent formula for the precession occurring in a bend element is to use the so-called “spin tune” in an electric field (derived in an appendix),

$$Q_s \equiv \left. \frac{d\alpha}{d\theta} \right|_E = G\beta^2\gamma - \frac{1}{\gamma} \stackrel{\text{also}}{=} \left(\frac{g}{2} - 1\right)\gamma - \frac{g/2}{\gamma}.$$

The precession advance in a bend element is then given by

$$\Delta\alpha = Q_s \Delta\theta,$$

where  $\Delta\theta$  is the total magnet bend angle. In our EDM ring, with all orbits having very nearly the magic velocity, the incremental precession in any single bend is very small.

## 12 Re-derivation to include horizontal betatron displacement

In estimating spin decoherence we need to account for the transverse position oscillations accompanying potential energy variation. For simplicity we assume the lattice is uniform, with no drift regions.

The spin precession angle  $\alpha$ , relative to the proton direction, evolves as

$$\frac{d\alpha}{dt} = \frac{eE(x)}{m_p c} \left( \frac{g\beta(x)}{2} - \frac{1}{\beta(x)} \right).$$

The variables  $\beta$ ,  $\gamma$ , and  $E$  in this equation are now being allowed to depend on  $x$ . Definition of angular momentum and its conservation yield

$$\frac{d\theta}{dt} = \frac{L}{\gamma m_p r^2};$$

(which is valid in bend regions, but would not be in drift regions, where  $r$  becomes ambiguous). In this equation the angular momentum  $L$  is a constant of the motion (because the force is radial) but  $\gamma$  and  $r = r_0 + x$  depend on  $x$ . Combining the two previous equations,

$$\frac{d\alpha}{d\theta} = \frac{eE(x)(r_0 + x)^2}{Lc\beta(x)} \left( \left( \frac{g}{2} - 1 \right) \gamma(x) - \frac{g/2}{\gamma(x)} \right),$$

- ▶ To find the evolution of  $\alpha$  over long times, for an individual particle, we need to average this equation.
- ▶ What makes this averaging difficult is the fact that the final factor has, intentionally, been arranged to cancel for the central, design particle.
- ▶ The initial factor, though not constant, varies over a quite small range. A promising approximation scheme for this factor is to neglect the (small) rapidly oscillating betatron contribution to  $x$  coming from the betatron oscillation, and retain only the off-momentum part  $x = D_x \Delta\gamma^0$  associated with the slowly varying synchrotron oscillation;

$$\left\langle \frac{d\alpha}{dt} \right\rangle (\gamma^0) = \frac{eE(D_x \Delta\gamma^0)(r_0 + D_x \Delta\gamma^0)^2}{Lc\beta(D_x \Delta\gamma^0)} \left\langle \left( \frac{g}{2} - 1 \right) \gamma(x) - \frac{g/2}{\gamma(x)} \right\rangle.$$

- ▶ No superscript “l” is needed on  $\gamma(x)$  in the final factor since only “inside” motion is under discussion.
- ▶ If the average of  $\langle \gamma \rangle$  were the inverse of  $\langle 1/\gamma \rangle$  the averaging over horizontal betatron oscillation would be easy. But this is not true.
- ▶ However the above factorization has allowed the averaging over  $\gamma^0$  to be deferred.

- ▶ The virial theorem can sometimes be used to obtain average behavior of multiparticle systems subject to central forces.
- ▶ Our application is complicated by the fact that fully relativistic mechanics has to be used.
- ▶ Also, though our electric field is centrally directed within any single deflecting element, because of drift regions in the lattice, the centers of the various deflection elements do not coincide.
- ▶ We can therefore calculate only the spin decoherence applicable to passage through the bend regions, which is where the overwhelmingly dominant part of the momentum evolution occurs.
- ▶ The independent variables  $\theta$  and  $t$  are very nearly, but not exactly proportional to each other instantaneously, so averages with respect to one or the other are not necessarily identically instantaneously.
- ▶ However, with bunched beams over long times,  $\theta$  and  $t$  are strictly proportional (on the average) and the two forms of averaging have to be essentially equivalent.
- ▶ Because there are so many variants of “the virial theorem” it is easier to derive it from scratch than to copy it from one of many possible references.

15 The “virial”  $G$  is defined, in terms of radius vector  $\mathbf{r}$  and momentum  $\mathbf{p}$ , by

$$G = \mathbf{r} \cdot \mathbf{p}$$

Our electric field is

$$\mathbf{E} = -E_0 \left( \frac{r_0}{r} \right)^{1+m} \hat{\mathbf{r}},$$

and Newton’s law gives

$$\frac{d\mathbf{p}}{dt} \equiv m_p \frac{d}{dt} (\gamma \mathbf{v}) = e\mathbf{E}.$$

In a bending element the time rate of change of  $G$  is given by

$$\begin{aligned} \left. \frac{dG}{dt} \right|_{\text{bend}} &= \dot{\mathbf{r}} \cdot \mathbf{p} + \mathbf{r} \cdot \dot{\mathbf{p}} \\ &= m_p \gamma v^2 - eE_0 \frac{r_0^{1+m}}{r^m} \\ &= m_p c^2 \gamma - m_p c^2 \frac{1}{\gamma} - eE_0 r_0 \frac{r_0^m}{r^m}. \end{aligned}$$

Averaging over time, presuming bounded motion, and therefore requiring  $\langle dG/dt \rangle$  to vanish, one obtains

$$\left\langle \frac{1}{\gamma} \right\rangle = \langle \gamma \rangle - \frac{E_0 r_0}{m_p c^2 / e} \left\langle \frac{r_0^m}{r^m} \right\rangle.$$

This provides the needed relation between  $\langle \gamma \rangle$  and  $\langle 1/\gamma \rangle$ .

16 Applying this result to perform the (time)-average yields

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle = \frac{eE(D_x \Delta\gamma^O)(r_0 + D_x \Delta\gamma^O)^2}{Lc\beta(D_x \Delta\gamma^O)} \left( -\langle \gamma \rangle + \frac{g}{2} \frac{E_0 r_0}{m_p c^2 / e} \left\langle \frac{r_0^m}{r^m} \right\rangle \right).$$

For specializing this result to frozen spin  $\gamma = \gamma_0$  operation, the following formulas, can be employed:

$$\begin{aligned} \gamma(x) &\equiv \gamma_0 + \Delta\gamma, \\ \frac{E_0 r_0}{m_p c^2 / e} &= \gamma_0 - \frac{1}{\gamma_0}, \\ \frac{r_0^m}{r^m} &\approx 1 - m \frac{x}{r_0}, \\ \gamma_0 &= \frac{g}{2} \left( \gamma_0 - \frac{1}{\gamma_0} \right). \end{aligned}$$

- ▶ These formulas assume the beam centroid energy and the storage ring lattice are exactly “magic”. If not true the average spin orientation would change systematically. What is being calculated is the spin orientation spreading.
- ▶ For perfectly sinusoidal synchrotron oscillations, the initial factor can be replaced by its average value. This yields

$$\left\langle \frac{d\alpha}{d\theta} \right\rangle \approx -\frac{E_0 r_0^2}{\beta_0 L c / e} \left( \langle \Delta\gamma^I \rangle + \frac{g}{2} \frac{m}{r_0} \left( \gamma_0 - \frac{1}{\gamma_0} \right) \langle x \rangle \right).$$

(The superscript “I” has been restored as a reminder that  $\Delta\gamma^I$  is evaluated within bend elements, as contrasted to within drift sections.) The numerical value of the leading factor is about 1.



- ▶ Second term on right hand side:
  - ▶ For  $m = 0$  the term vanishes identically.
  - ▶ For  $m = \pm 0.002$  in successive bends, the term proportional to  $m$  also cancels, leaving only a term of order  $m^2$ .
  - ▶ The betatron-average  $\langle x \rangle$  would, itself vanish, at least to the (extremely accurate) extent to which the lattice is linear.
- ▶ First term on right hand side:
  - ▶ The term  $\Delta\gamma'$  on the right hand side depends sensitively on  $m$ . But, for the WW-AG-CF lattice, the full range of  $\Delta\gamma'$  for captured particles is  $10^{-7}$ . (See graph from earlier lecture.) Furthermore there is cancellation to leading order in bends with alternating  $m$  values.

- ▶ We have shown, therefore, for the WW-AG-CF lattice, that decoherence in the bend regions can be neglected even in the presence of horizontal betatron oscillations,
- ▶ We have previously argued (though perhaps not yet persuasively enough) that decoherence associated with vertical betatron oscillation can also be neglected.
- ▶ Numerical simulation results obtained by Yannis and Selcuk have been consistent with this analysis.
- ▶ Furthermore, very long spin coherence times have been demonstrated for deuterons in the COSY storage ring in Juelich, Germany, though only after quite delicate adjustment of nonlinear elements in the ring. Nevertheless COSY is a strong focusing ring for which spin decoherence can be expected to be far greater than in our weak focusing WW-AG-CF lattice.
- ▶ If beam bunches can survive for days their polarization states can probably survive as well.

## 19 Spin evolution in idealized lattice

*Except for idealized specializations, this section reproduces Sections II, III, and IV of Chapter 2 of S.Y. Lee's book "Spin Dynamics and Snakes in Synchrotrons", almost line by line. My purpose is to demonstrate the spectacular simplification that results when the design orbit can be assumed to remain in a single plane. A tiny, but confusing, further alteration is to use  $y$  (rather than Lee's  $z$ ) as vertical coordinate. The coordinates  $(x, s, y)$ , in (horizontal, longitudinal, vertical) order, (which for Lee is  $(x, s, z)$ ) will be retained, in spite of the fact that more conventional in the storage ring world would be (horizontal, vertical, longitudinal). This may seem very confusing (partly because the longitudinal Frenet coordinate is to be  $s$ ) but avoids mis-interpreting  $z$  as longitudinal, and it means that, to convert any Lee formula to a formula in this appendix, one only needs to make the replacement  $z \rightarrow y$  (plus simplifying assumptions). What motivates this coordinate order for spin evolution is that it is useful for the first two coordinates  $(x, s)$  as 2D coordinates in the (horizontal) ring design plane, with  $y$  being vertical.*

## 20 Spinor formalism

For simplicity in copying from Lee, the discussion will be limited to ordinary (magnetic) rings. The fundamental BMT spin evolution formula is (SYL-2.40)

$$\frac{d\mathbf{S}}{d\theta} = \mathbf{S} \times \mathbf{F},$$

where  $\theta$  is bend angle with  $d\theta = ds/\rho$  in a dipole,  $\rho$  is radius of curvature, and the applied torque (multiplied by the appropriate factor) is (SYL-2.41)

$$\mathbf{F} = F_x \hat{\mathbf{x}} + F_s \hat{\mathbf{s}} + F_y \hat{\mathbf{y}}.$$

In our idealized ring the design orbit lies in the  $(x, s)$  plane and  $(y = 0, y' = 0, y'' = 0)$ , where derivatives with respect to  $s$  are indicated by primes. Using these values, and copying from (SYL-2.42), the components of  $\mathbf{F}$  are

$$\mathbf{F} = \begin{pmatrix} F_x \\ F_s \\ F_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(1 + G\gamma) + (1 + G\gamma)\rho x'' \end{pmatrix}.$$

S.Y. justifies dropping the  $x''$  factor from  $F_y$  by stating that its average (presumably over betatron oscillation) is zero. The validity of this simplification will be returned to shortly.

Expressing  $\mathbf{S}$  in terms of its components (SYL-2.43),

$$\mathbf{S} = S_x \hat{\mathbf{x}} + S_s \hat{\mathbf{s}} + S_y \hat{\mathbf{y}},$$

and using  $d\hat{\mathbf{x}}/d\theta = \hat{\mathbf{s}}$  and  $d\hat{\mathbf{s}}/d\theta = -\hat{\mathbf{x}}$  we get (SYL-2.44)

$$\frac{d}{d\theta} \begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix} = \begin{pmatrix} (1+F_y)S_s \\ -(1+F_y)S_x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & (1+F_y) & 0 \\ -(1+F_y) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} S_x \\ S_s \\ S_y \end{pmatrix}.$$

This amounts to being the BMT equation, in S.Y. Lee notation, under our 2D assumptions. Continuing to follow Lee, we define (complex) spin components

$$S_{\pm} = S_x \pm iS_s, \quad \text{with inverses} \quad S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - S_-}{2i},$$

as well as field components  $F_{\pm} = F_x \pm iF_s$  (that will not actually be needed under our special assumptions) to obtain (averaged) equations of motion,

$$\frac{d}{d\theta} \begin{pmatrix} S_{\pm} \\ S_y \end{pmatrix} = \begin{pmatrix} \pm iG\gamma S_{\pm} \\ 0 \end{pmatrix}.$$

Solving these differential equations, the spin evolution is given by (SYL-2.45);

$$S_{\pm} = e^{\pm iG\gamma\theta} S_{\pm 0}, \quad S_y = \text{constant}.$$

- ▶ One sees that  $S_{\pm}$  are eigenfunctions of the propagation, with eigenvalues  $e^{\pm iG\gamma\theta}$ .
- ▶ Introduction of  $S_{\pm}$  has further decoupled the spin motion.
- ▶ For propagation around the full ring, one sets  $\theta = 2\pi$  and obtains  $e^{\pm 2\pi iG\gamma}$  as the eigenvalues. It is because the eigenvalues are complex, while the  $S_i$  components are real, that the expansions of eigenfunctions as superpositions of  $S_i$  components has required a complex coefficient.
- ▶ On the basis of this solution, for magnetic rings, the spin tune of an “ideal” lattice has been shown to be

$$Q_s = G\gamma.$$

- ▶ A curious feature of this relation is its lack of dependence on radius of curvature  $\rho$ . This came about during averaging over betatron oscillations, when  $x''$  was dropped.
- ▶ Ordinarily the radius of curvature  $\rho$  has roughly the same value in every bend element.
- ▶ But one could design a ring where  $\rho$  had different values in different sectors. This would leave relation  $Q_s = G\gamma$  unaffected. To say, therefore, that  $Q_s = G\gamma$  is a property of an “ideal” lattice, has significantly expanded the meaning of “ideal”.
- ▶ A (horizontally) misaligned quadrupole steers the central orbit, but leaves the central closed orbit in the same (horizontal) design plane. Why don't we just pretend that the misalignment is part of the “design”, in which case we still have  $Q_s = G\gamma$ .

- ▶ It seems to me, therefore, that  $G\gamma$  is a global invariant, unaffected by lattice errors, to the extent they leave the central closed orbit in a single plane. (Operationally one always achieves this condition to quite high accuracy.)
- ▶ It seems, therefore, that the virial theorem argument can be used to justify S.Y. Lee's earlier dropping of the  $x''$  factor.
- ▶ Small vertical betatron motion would not alter this conclusion to lowest order. Non-commutation of rotations could violate the averaging to zero, but only proportional to the product of already extremely small amplitudes.
- ▶ All this is consistent with the earlier 2D virial theorem demonstration that spin decoherence can be neglected in our ultraweak focusing WW-AG-CF lattice. Once a beam bunch has been "captured" its total spin precesses almost as if it were a single particle



## 25 Spinor representation of spin evolution

Continuing to specialize formulas from S.Y. Lee, our spin evolution equation has decoupled to two equations (SYL-2.46)

$$\begin{aligned}\frac{d\mathbf{S}_{\parallel}}{d\theta} &= G\gamma \hat{\mathbf{y}} \times (S_x \hat{\mathbf{x}} + S_s \hat{\mathbf{s}} + S_y \hat{\mathbf{y}}) = G\gamma (S_x \hat{\mathbf{s}} - S_s \hat{\mathbf{x}}), \\ \frac{dS_y}{d\theta} &= 0.\end{aligned}$$

Instead of the 3-component real vectors used so far, one can represent the spin by a 2-complex component general spinor (SYL-2.51),

$$\Psi = \begin{pmatrix} u \\ d \end{pmatrix},$$

Introducing the Pauli matrices, as the three components of a “vector”  $\boldsymbol{\sigma}$ ,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_s = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the 3D spin components can be expressed in terms of  $\psi$  as (SYL-2.47)

$$S_i = \Psi^\dagger \sigma_i \Psi, \quad \text{for } i = x, s, y.$$

- ▶ This permits the  $S_1$  components to be expressed in terms of  $u$  and  $d$ , (SYL-2.52);

$$S_1 = u^*d + ud^*, \quad S_2 = -i(u^*d - ud^*), \quad S_3 = |u|^2 - |d|^2.$$

- ▶ Even though  $u$  and  $d$  are complex in general, the  $S_i$  components are always real. With this notation, the spin evolution equation becomes (SYL-2.48)

$$\frac{d\Psi}{d\theta} = -\frac{1}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{\Psi}) = -\frac{i}{2}\mathbf{H}\Psi, \quad \text{where } \mathbf{H} = \begin{pmatrix} G\gamma & 0 \\ 0 & -G\gamma \end{pmatrix} = G\gamma\sigma_y.$$

- ▶ Notice that Lee's perturbation function (SYL-2.49),  $\xi(\theta) = F_x(\theta) - iF_s(\theta)$ , which would have appeared in the off-diagonal elements of  $\mathbf{H}$ , actually vanishes identically (as a consequence of our pure-planar design orbit assumption).
- ▶ For a lattice with arbitrary errors the full formalism continues to be valid, in which case, of course, the off-diagonal elements of  $\mathbf{H}$  need to be correctly included.

## 27 Spinor transfer matrix

- ▶ For initial polarization state  $\Psi(\theta_1)$ , solving this differential equation produces a later polarization state  $\Psi(\theta_2)$  given by (SYL-2.54)

$$\Psi(\theta_2) = e^{-\frac{i}{2} \mathbf{H}(\theta_2 - \theta_1)} \Psi(\theta_1) \equiv t(\theta_2, \theta_1) \Psi(\theta_1).$$

which defines  $t(\theta_2, \theta_1)$  as a “transfer matrix” in the spinor formalism.

- ▶ With  $\mathbf{H}$  diagonal, the spin evolves as

$$t(\theta_2, \theta_1) = e^{-\frac{i}{2} G\gamma(\theta_2 - \theta_1)\sigma_y}.$$

- ▶ This evolution formula resembles the earlier  $S_{\pm}$  eigenfunction evolution formula closely, but with the important difference that the matrix  $\sigma_y$  appears here in the exponent. This does not seriously complicate algebraic manipulations for which this result is to be applied.
- ▶ All this “looks like” quantum mechanics, but it has just been mathematical manipulation.

## 28 One turn map (OTM)

- ▶ Also useful is the one turn transfer map (OTM),  $t(\theta)$  from arbitrary initial angle  $\theta$ , once around the ring and back to the same location.
- ▶ It satisfies (SYL-2.58)

$$\Psi(\theta + 2\pi) = \prod_{j=1}^N t(\theta_{j+1}, \theta_j) \Psi(\theta), = t(\theta) \Psi(\theta),$$

which concatenates the transfer maps over all  $N$  ring sectors starting from  $\theta$  and returning to the same position.

## 29 Storage ring as “Penning-Like Trap” —motivation

- ▶ The possibility of storing a large number, such as  $10^{10}$ , of identically polarized particles makes a storage ring an attractive charged particle “trap”.
- ▶ But, compared to a table top trap, a storage ring is a quite complicated assemblage of many carefully, but imperfectly, aligned components, powered from not quite identical sources.
- ▶ Fortunately, particle magnetic dipole moments (MDM) have been measured to exquisitely high precision. For our purposes MDM’s can be treated as exactly known.
- ▶ High enough beam polarization, and long enough spin coherence time SCT, have made it possible to “freeze” the spins.
- ▶ In this frozen state, the importance of some inevitable machine imperfections, that might otherwise be expected to dominate the errors, is greatly reduced. Examples are beam energy spread and ring element positioning and alignment uncertainties.

- ▶ (With the benefit of RF-imposed synchrotron oscillation stability) the average beam energy is fixed with the same exquisitely high accuracy with which the MDM is known.
- ▶ The polarization vector serves as the needle of a perfect speedometer. With the RF frequency also known to exquisite accuracy, the revolution period is similarly well known.
- ▶ Then, irrespective of element locations and powering errors, the central orbit circumference is, if not perfectly known, at least known to be constant in time (except for knowable and controllable element changes).
- ▶ To encompass all of these considerations the storage ring can be referred to as a “polarized beam trap”.
- ▶ This strategem reduces the importance of some sources of error, but without eliminating them altogether. Of course one will build the EDM storage ring as accurately as possible.
- ▶ The lack of concern about element absolute positioning must not to be confused as lack of concern for BPM, orbit positioning precision, even assuming the ring has been tuned to be a perfect trap.
- ▶ Reduction of systematic EDM measurement error will depend critically on precision beam positioning control.

## 31 Small deviations from magic condition

- Suppose the beam magnetization phase has been locked by external feedback, for example in an electron ring. One can then take advantage of the precisely-known electron magnetic moment  $\mu_e$  and anomalous moment  $G_e$ . The spin tune  $Q_s^E$  relates to precession around the vertical axis. In an all-electric ring  $Q_s^E$  is given by

$$Q_s^E = G_e \gamma - \frac{G_e + 1}{\gamma},$$

For frozen spin electrons at the “magic” value,  $Q_s^E = 0$ ,  $\gamma = \gamma_m$ , where

$$\gamma_m = \sqrt{\frac{G_e + 1}{G_e}} = 29.38243573.$$

Solving for  $\gamma$ , (and requiring  $\gamma > 0$ ),

$$\gamma = \frac{Q_s^E + \sqrt{Q_s^{E2} + 4G_e(G_e + 1)}}{2G_e},$$

we then obtain

$$\gamma = 29.38243573 + 431.16379 \frac{\Delta f_y}{f_0} + 3163.5 \left( \frac{\Delta f_y}{f_0} \right)^2 + \dots$$

Here  $Q_s^E$  has been re-expressed in terms of the frequency deviation from magic,  $\Delta f_y = f_y - f_m$ , of the polarization around a vertical axis. This formula is intended for use only near  $\gamma_m$ , with the ratio  $\Delta f_y/f_0$  being a tiny number, less than  $10^{-5}$  for example.

## 32 Pseudo-frozen spin magnetic ring parameters for e, p, d and He<sup>3</sup>

- ▶ Because the EDM effect is so small, one seeks a configuration in which the EDM torque is being applied almost all the time, and its effect is monotonic—constructively building up the EDM-induced precession.
- ▶ Ideally the EDM torque present in each of the bending elements in the ring would have this property. (Otherwise any EDM accumulation would be limited to a small sector of the ring, and would be proportionally weaker.)
- ▶ Fully constructive accumulation would require the spin tune to vanish. This is a necessary, but not necessarily sufficient, condition for fully constructive EDM accumulation.
- ▶ The next figure provides plots of magnetic lattice tunes for protons, deuterons, helions, and electrons, with integer and half integer crossings emphasized. There is no case for which the spin tune vanishes. This prevents any EDM-induced precession occurring in the bending magnets from accumulating constructively to give an EDM signal proportional to the run duration.
- ▶ The table shows some of the pseudo-frozen spin possibilities for the four particle types.



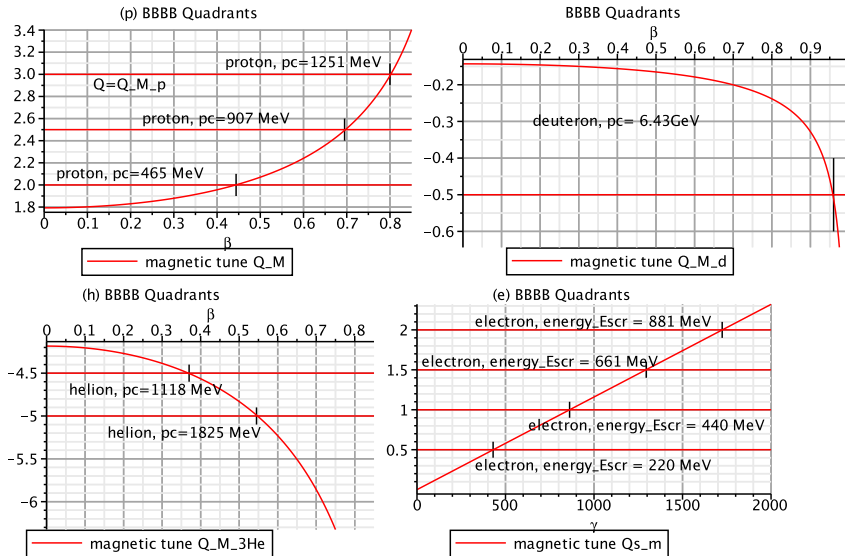


Figure: Magnetic lattice spin tunes for protons, deuterons, helions, and electrons.

**Table:** Candidate magic energies for magnetic ring pseudo-frozen spin operation. Particles considered are protons, deuterons, helium-3, and electrons. OTM is “one-turn-map” and STM is “spin transfer matrix”.  $Q_{s0}$  is the uncorrected spin tune (read from Figure 1).  $N_S$  is a candidate number of equal-strength solenoidal snakes (of arbitrary sign) capable of adjusting the spin tune to zero.

	$pc$ or $E$		unit	$Q_{s0}$	OTM	solenoid $\pi$ -snake strength, [T-m]	
p	465	$pc$	MeV	2.0		4.88	
	907			2.5		9.52	
	1251			3.0		13.12	
d	6.43	$pc$	GeV	-0.5		57.7	
He3	1118	$pc$	MeV	-4.5		18.6	
	1825			-5.0		30.4	
e	220	$E$	MeV	0.5	$RSR, (RSR)^2$	2.31	
	440			1.0		$RSR, (RSR)^2$	4.62
	661			1.5			6.93
	881			2.0			9.24

## 35 Magnetic ring spin tune alteration using Siberian snakes

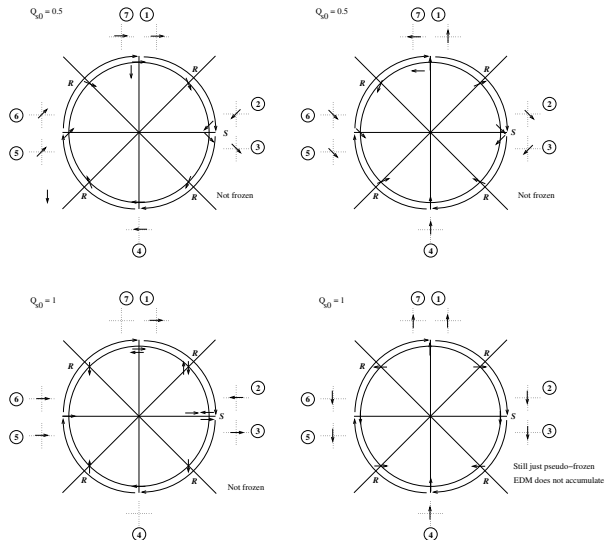
- ▶ Question: can Siberian snakes included in the ring can have the property of shifting the spin tune to zero?
- ▶ A perfect solenoidal Siberian snake reflects a spin vector already in the horizontal plane into another vector in the same plane. Operationally this reflection can be tuned to be “perfect”, at least averaged over long times.
- ▶ This alters the apparent spin tune, perhaps producing  $Q_s = 0$ .
- ▶ The following figures show that this can, in fact, be accomplished. So  $Q_s = 0$  can be achieved in a magnetic lattice with Siberian snakes.
- ▶ Unfortunately, subsequent figures will show, that monotonic EDM-induced precession still does not occur. Since each snake reverses the accumulated EDM precession, it is clear that an even number of snakes is required. When the snakes are uniformly distributed the spins can be frozen, but there is no net EDM effect per turn. And with the snakes non-uniformly distributed the spins cannot be frozen.

### 36 Can pseudo-frozen spin satisfy $Q_s = 0$ in magnetic ring? Yes.

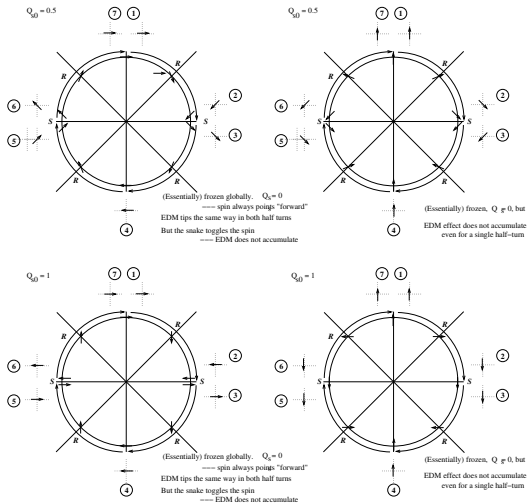
- ▶ From the previous figures it can be seen that there are no examples for which the spin tune vanishes in a magnetic ring. One can investigate whether Siberian snakes can be used to produce truly global frozen spin operation in a magnetic ring such as COSY, in spite of the fact that there are no examples for which the spin tune vanishes.
- ▶ Can one design a magnetic ring that includes (longitudinal) solenoidal Siberian snakes and has the property that a particle starting with spin pointed in the forward hemisphere, remains for ever with spin pointed into the forward hemisphere. In other words, can the spin tune be arranged to vanish?. The effect of any non-zero EDM would tend then, always, to tip the spin out of the plane, say up, as it passes through each bending magnet.
- ▶ Figures below show that  $Q_s = 0$  can, in fact, be achieved using Siberian snakes.

## 37 Can snakes enable EDM measurement in magnetic ring? No.

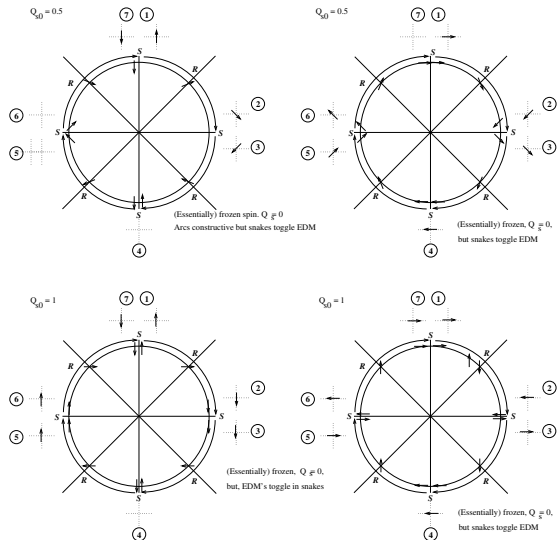
- ▶ Regrettably, Figures 2, 3, and 4 with, respectively, one, two, or four snakes, show that the Siberian snakes invariably have the effect of flipping any pre-existing EDM-induced spin precession. This prevents any systematic accumulation of EDM signal over multiple turns around any purely magnetic storage ring.
- ▶ By superimposing electric and magnetic it is theoretically easy to produce a frozen spin lattice for any charged fundamental particle without using Siberian snakes. As a practical matter though, it is extremely difficult, experimentally, to superimpose the required extremely strong electric field on even a weak magnetic field.
- ▶ To avoid this impediment one can contemplate a ring in which some sectors are magnetic, and others electric. Regrettably, this option is probably not satisfactory. Inevitably, element placement and powering errors introduce spin precession errors. To the extent all precessions leave the spins in the same horizontal plane separating electric and magnetic bending would probably be satisfactory. because rotations about the same axis commute, and these spurious rotations would tend to cancel.
- ▶ But precessions around different axes. When passing through, say, an electric bend sector, three significant precessions will have accumulated, a large spin tune precession, a small element error precession, and a small EDM-induced precession. Failure of commutation of successive rotations will tangle these precessions enough to make it impossible to isolate the desired EDM contribution.



**Figure:** Some single snake possibilities for frozen spin modification of a magnetic ring using helical snakes. Snake locations are indicated by "S", natural spin tune precession by "R". Even in cases with the spins (essentially) frozen,  $Q_s = 0$ , the snake toggles any EDM precession, which prevents the build-up of any EDM signal.

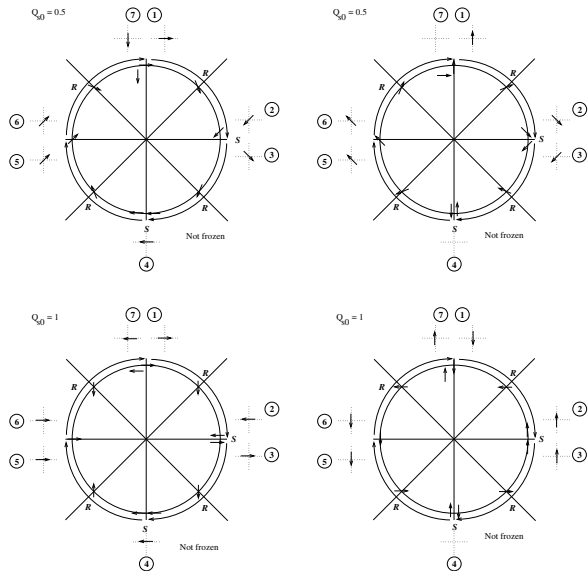


**Figure:** Some two snake possibilities for frozen spin modification of a magnetic ring using helical snakes. Snake locations are indicated by "S", natural spin tune precession by "R". For  $Q_s = 1$ , with no snakes, an initially forward-pointed spin would only be "pseudo-frozen", pointing forward again at the origin, but also half way around the ring.  $Q_s = 0.5$ , with no snakes, an initially forward-pointed spin, precesses through  $\pi$  after one turn, and is forward-pointed again after two turns. Even in cases with snakes (essentially) freezing the spins to  $Q_s = 0$ , the snakes toggle any EDM precession, which prevents the build-up of any EDM signal.



**Figure:** Some four snake possibilities for frozen spin modification of a magnetic ring using helical snakes. Snake locations are indicated by "S", natural spin tune precession by "R". Even in cases with the spins (essentially) frozen,  $Q_s = 0$ , the snakes toggle any EDM precession, which prevents the build-up of any EDM signal.





**Figure:** To prevent toggling of the EDM signal there must be an even number of snakes. Furthermore they must be "unbalanced" as in this figure. However, since there is no frozen spin case, there is no EDM candidate.



J.D. Jackson, *Classical Electrodynamics*, 3rd edition, John Wiley, 1999



S.Y. Lee, *Spin Dynamics and Snakes in Synchrotrons* World Scientific, 1997