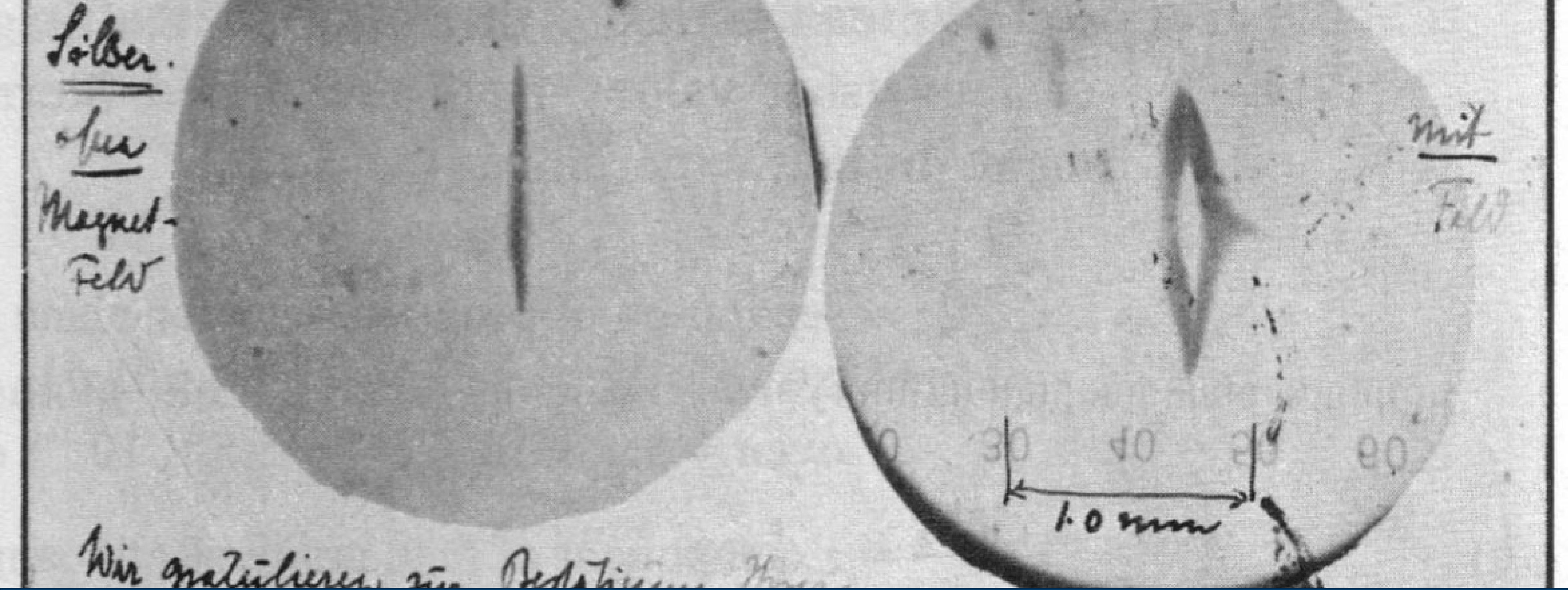

POLARIMETRY

An Introduction into Polarimetry and the Calculation of the Figure of Merit

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OUTLINE

- (short) History of Spin
- Polarization
- From Cross Section to Polarization
- Cross Ratio Methods
- Statistical Error & Figure of Merit
  - FoM: Counting Method
  - FoM: Weighting Method & Optimal Weight
  - FoM: Extended Log-Likelihood
- FoM Comparison on a WASA Dataset

(SHORT) HISTORY OF SPIN

Spin – Theoretical & Experimental Approaches

- 1922: Otto Stern & Walther Gerlach performed their famous experiment where they observed “space-quantization”
- 1924: Electron spin was introduced by Wolfgang Pauli as a “two-valued quantum degree of freedom”, needed to formulate the *Pauli exclusion principle* to explain the hydrogen spectrum
- 1927: Pauli formalized the theory of spin using the modern theory of quantum mechanics formulated by Heisenberg and Schödinger
- 1928: Relativistic extension of the theory of spin by Paul Dirac
- 1932: First Measurements of electron polarization using double-scattering experiments (Nevill Francis Mott)
- 1940: Pauli proved the spin-statistics-theorem: Fermions have a half-integer spin and Bosons have a integer spin
- ~ 1950: First polarization measurements for protons using double-scattering experiments

POLARIZATION

Definitions

Vector Polarization for Protons

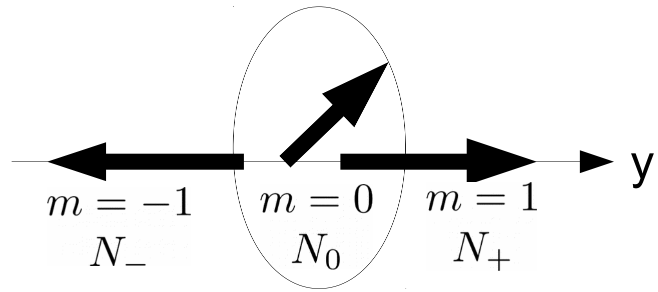
$$P_y = \frac{N_+ - N_-}{N_+ + N_-} = p_+ - p_-$$

Intensity $I = N_+ + N_-$

Probability $p_+ = \frac{N_+}{I}$, $p_- = \frac{N_-}{I}$

Vector Polarization for Deuterons

$$P_y = \frac{N_+ - N_-}{N_+ + N_0 + N_-} = p_+ - p_-$$



Tensor Polarization for Deuterons

$$P_{yy} = \frac{N_+ - 2N_0 + N_-}{N_+ + N_0 + N_-} = 1 - 3p_0$$

Intensity $I = N_+ + N_0 + N_-$

Probability $p_+ = \frac{N_+}{I}$, $p_- = \frac{N_-}{I}$, $p_0 = \frac{N_0}{I}$

POLARIZATION

Measurement from Elastic Scattering

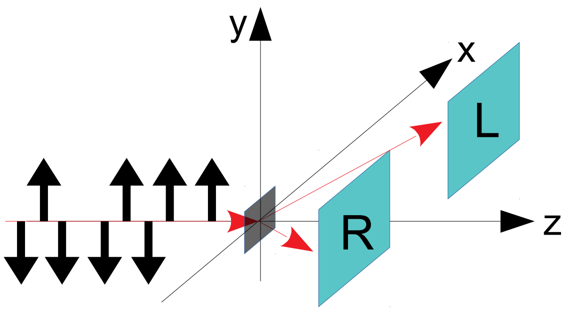
Θ	Polar Angle
Φ	Azimuthal Angle
$\sigma_0^{p,d}$	Unpol. CS
$A_y^{p,d}$	Vector Analyzing Power

The total elastic cross section for protons:

$$\sigma_p(\theta,\phi) = \sigma_0^p(\theta)(1 + P_y A_y^p(\theta)\sin(\phi))$$

And for (only vector polarized) deuterons:

$$\sigma_d(\theta,\phi) = \sigma_0^d(\theta)(1 + \frac{3}{2} P_y A_y^d(\theta)\sin(\phi))$$



Assuming the same polarization and acceptance in both detectors:

$$n^{\pm}(\theta) = \alpha(\theta)(1 \pm P\beta(\theta))$$

The number of hits for spin up/down or detector left/right respectively, for a given polar angle

α Contains all the information about unpol. CS, luminosity and acceptance

β Contains the information about the vector analyzing power

POLARIZATION

Measurement from Elastic Scattering

Using the definition of $n^{\pm}$ an expression for the expected value of N can be given

$$\langle N^{\pm} \rangle = (1 \pm P \langle \beta \rangle) \int \alpha(\theta) \partial\theta$$

With the following probability density functions (pdf):

Unpolarized pdf: $n^0 = \frac{\alpha}{\int \alpha \partial\theta}$ Polarized pdf: $n^{\pm} = \frac{\alpha(1 \pm P\beta)}{\int \alpha(1 \pm P\beta)\partial\theta}$

Using this pdf's the following expressions can be deduced

$$\Rightarrow \langle \beta \rangle = \frac{\int \beta\alpha\partial\theta}{\int \alpha\partial\theta} \approx \frac{\sum^{+} \beta_i + \sum^{-} \beta_i}{\langle N^{+} \rangle + \langle N^{-} \rangle} \quad \text{with} \quad \beta_i = \beta(\theta_i)$$

$$\Rightarrow P = \frac{1}{\langle \beta \rangle} \frac{\langle N^{+} \rangle - \langle N^{-} \rangle}{\langle N^{+} \rangle + \langle N^{-} \rangle} \approx \frac{\langle N^{+} \rangle - \langle N^{-} \rangle}{\sum^{+} \beta_i + \sum^{-} \beta_i}$$

POLARIZATION

Measurement from Elastic Scattering

As an estimator for the polarization, one finds:

$$\hat{P} = \frac{N^+ - N^-}{\sum^+ \beta_i + \sum^- \beta_i}$$

unweighted

$$\hat{P}_\omega = \frac{\sum^+ \omega_i - \sum^- \omega_i}{\sum^+ \omega_i \beta_i + \sum^- \omega_i \beta_i}$$

weighted with an arbitrary weight $\omega_i = \omega(\theta_i)$

It can be shown that $\left\langle \hat{P}_\omega \right\rangle = \left\langle \hat{P} \right\rangle = \langle P \rangle = P \quad \forall \omega$

The expected value for the polarization is independent on the weight function!

FIGURE OF MERIT (FOM)

Definition

Applying the statistical error propagation to the weighted estimator yields:

$$\sigma^2(\hat{P}_\omega) = \frac{\left\langle \omega^2 \right\rangle - P^2 \left\langle (\omega\beta)^2 \right\rangle}{\langle N \rangle \left\langle \omega\beta \right\rangle^2}$$

With the definition for the Figure of Merit: $\mathrm{FoM}(\hat{P}_\omega) = \sigma^{-2}(\hat{P}_\omega)$

As the quantity that minimizes the statistical error, one gets:

$$\mathrm{FoM}(\hat{P}_\omega) = \langle N \rangle \frac{\langle \omega\beta \rangle^2}{\langle \omega^2 (1 - P^2\beta^2) \rangle}$$

The statistical error of the polarization does depend on the weight function!

FIGURE OF MERIT (FOM)

For a Unweighted Counting Method

By choosing a weight $\omega = 1$, the following expression for the FoM will be found:

$$\mathrm{FoM}(\hat{P}) = \langle N \rangle \frac{\langle \beta \rangle^2}{1 - P^2 \langle \beta^2 \rangle}$$

With the assumption that $P\beta \ll 1$ the formula can be reduced to:

$$\mathrm{FoM}(\hat{P}) \approx \langle N \rangle \langle \beta \rangle^2$$

FIGURE OF MERIT (FOM)

Improved Weighting Method

Finding the optimal weight ω for the FoM can achieved using calculus of variations

$\omega = \omega_0 + \epsilon\eta$ with

- ω_0 Optimal weight
- $\epsilon \ll 1$ Small deviation from the optimum
- η With an arbitrary function

From the condition $$\left.\frac{\partial \mathrm{FoM}(\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = 0 \quad \Rightarrow \omega_0 = \frac{\beta}{1 - P_0^2\beta^2}$$

$$\Rightarrow \mathrm{FoM}(\hat{P}_{i\omega}) = \langle N \rangle \frac{\left\langle \frac{\beta^2}{1-P_0^2\beta^2} \right\rangle^2}{\left\langle \beta^2 \frac{1-P^2\beta^2}{(1-P_0^2\beta^2)^2} \right\rangle}$$

Can be simplified using:
$P_0 \approx P$ and $P\beta \ll 1$

$$\mathrm{FoM}(\hat{P}_{i\omega}) \approx \langle N \rangle \frac{\langle \beta^2 \rangle}{1 - P^2 \frac{\langle \beta^4 \rangle}{\langle \beta^2 \rangle}} \approx \langle N \rangle \langle \beta^2 \rangle$$

FIGURE OF MERIT (FOM)

FoM from the Extended Log-Likelihood Method

Different approach for an estimator for the polarization P: Extended Log-Likelihood

$$p^{\pm}_{P,\langle N^{\pm}\rangle}(\alpha_i,\beta_i) = \frac{a_i(1 \pm P\beta_i)}{\langle N^{\pm}\rangle}$$

Probability of finding (α_i, β_i) for given parameters $P, \langle N^{\pm}\rangle$

$$L_{\alpha,\beta}(P,\langle N^{\pm}\rangle) = \prod_i^{\langle N^+\rangle} \frac{\alpha_i(1+P\beta_i)}{\langle N^+\rangle} \prod_i^{\langle N^-\rangle} \frac{\alpha_i(1-P\beta_i)}{\langle N^-\rangle}$$

$\langle N^{\pm}\rangle$ Poisson-distributed $\left(p(k,\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}\right) \Rightarrow$ *Extended Likelihood Function*

$$L^{Ex}_{\alpha,\beta}(P) = p(N^+, \lambda = \langle N^+\rangle)p(N^-, \lambda = \langle N^-\rangle)L_{\alpha,\beta}(P, N^{\pm})$$

$$\mathcal{L}^{Ex}_{\alpha,\beta}(P) = \sum_i^{N^{\pm}} ln(1 \pm P\beta_i) + \sum_i^{N^{\pm}} ln(\alpha_i) - 2\int \alpha \partial\theta - ln(N^+!N^-!)$$

independent of P !

FIGURE OF MERIT (FOM)

FoM from the Extended Log-Likelihood Method

Finding an estimator from the Extended Log-Likelihood-Function

$$\frac{\partial \mathcal{L}^{Ex}_{\alpha,\beta}(P)}{\partial P} \stackrel{!}{=} 0 = \sum^{N^+} \frac{\beta_i}{1 + P\beta_i} - \sum^{N^-} \frac{\beta_i}{1 - P\beta_i}$$

Assuming $P\beta < 1$ a Taylor-Approximation can be used: $\frac{\beta}{1 \pm P\beta} \approx \beta \mp P\beta^2$

$$\hat{P}_{\mathcal{L}} = \frac{\sum^+ \beta_i - \sum^- \beta_i}{\sum^+ \beta_i^2 + \sum^- \beta_i^2}$$

Using the Fisher-Theorem $\sigma^{-2}(\hat{P}_{\mathcal{L}}) = -\frac{\partial^2 \mathcal{L}^{Ex}_{\alpha,\beta}(P)}{\partial P^2}$ one finds:

$$\mathrm{FoM}(\hat{P}_{\mathcal{L}}) = \langle N \rangle \left\langle \frac{\beta^2}{1 - P^2\beta^2} \right\rangle \approx \langle N \rangle \left\langle \beta^2 \right\rangle$$

POLARIZATION & FOM

Intermediate Summary

Finding an estimator for the polarization leads to the following results:

$$\hat{P}_\omega = \frac{\sum^+ \omega_i - \sum^- \omega_i}{\sum^+ \omega_i \beta_i + \sum^- \omega_i \beta_i}$$

$\hat{P}$ for $\omega = 1$ Counting Method

$\hat{P}_{i\omega}$ for $\omega = \frac{\beta}{1-P_0^2\beta^2}$ Optimal Weight

$\hat{P}_{\mathcal{L}}$ for $\omega = \beta$ Extended Log-Likelihood

But the selected weight does not influence the expected value of the polarization!

$$\left\langle \hat{P}_\omega \right\rangle = P \quad \forall \omega$$

The weight influence the statistical error of the polarization: $\sigma^{-2}(\hat{P}) \equiv \mathrm{FoM}(\hat{P})$

$$\mathrm{FoM}(\hat{P}) \approx \langle N \rangle \langle \beta \rangle^2$$

Counting Method

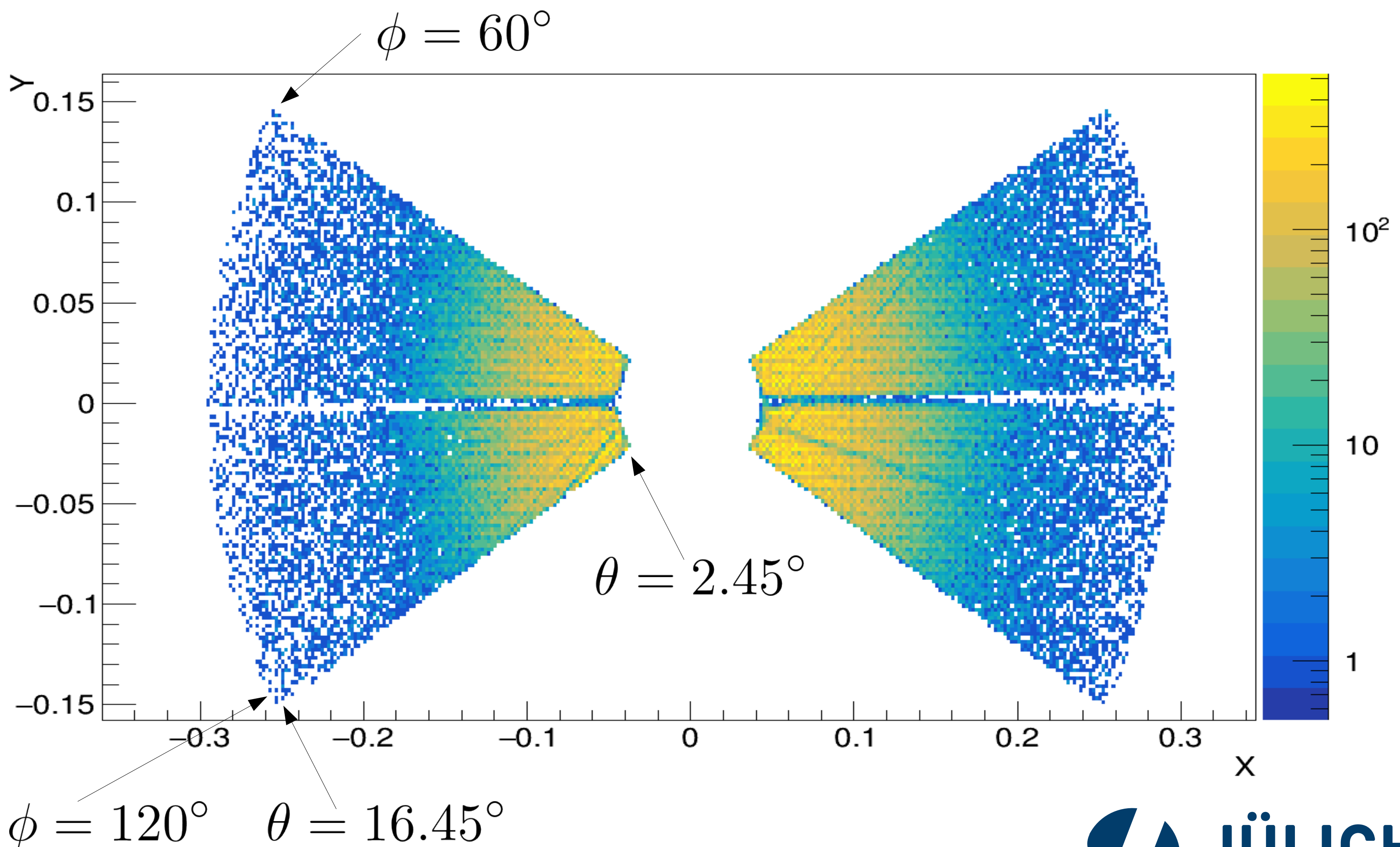
$$\mathrm{FoM}(\hat{P}_{i\omega}) = \mathrm{FoM}(\hat{P}_{\mathcal{L}}) \approx \langle N \rangle \langle \beta^2 \rangle$$

Optimal Weight / Extended Log-Likelihood

FoM can be improved by choice of weight. $\langle \beta^2 \rangle \geq \langle \beta \rangle^2$

FOM ON A WASA DATASET

Hit Distribution for 270 MeV Deuterons with Spin-Up Polarization



FOM ON A WASA DATASET

FoM Calculated for 270 MeV Deuterons using Satou's Analizing Power