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Review

Spin response formalism in circular accelerators

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ABSTRACT

We present the principal features of the so-called "spin response formalism", which is linear response theory applied to the spin dynamics in circular accelerators. The formalism is useful for calculating the resonance strengths of several classes of first-order spin resonances in rings, including those for spin flippers. We describe some of the successful applications of the formalism to various storage rings. We include a brief comparison with other formalisms and indicate topics for future work.

Contents

1. Introduction

The spin response formalism [1] is linear response theory applied to the spin dynamics of polarized beams in circular accelerators. The formalism provides a systematic way to calculate the strengths of several classes of spin resonances, via so-called "spin response functions". An early version of a spin response function first appeared in Ref. [2], where it was denoted by F and was not explicitly connected to linear response theory. The function F was rederived in Ref. [3], using linear response theory. The fully general theory was formulated in Ref. [1], to derive the full set of spin response functions F_1-F_5 (to be defined later in this paper) in a unified treatment. The above function F corresponds to F_3 in the terminology of Ref. [1]. To date, however, the major reference to the formalism is in Russian [1] (see also a limited description in English in Ref. [4]). This paper aims to describe the principal details of the formalism in Ref. [1] in an English-language paper, to make the spin response formalism more widely accessible to workers in the field.

The first spin flipper was operated at VEPP-2M in 1980 (announced internationally in 1982 [5]), and much of the early experimental and theoretical development of spin flippers was carried out at the Budker Institute for Nuclear Physics. Additionally, formulas to calculate resonance strengths for polarized beams in storage rings, including spin flippers, were also developed (e.g. see Refs. [2,3,6]). The

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spin response formalism is a generalization or unification of ideas from these earlier works, to develop a systematic procedure to calculate resonance strengths for several classes of spin resonances, using linear response theory.

As recent experimental measurements have demonstrated, the spin response formalism is not only a practical quantitative tool for the calculation of resonance strengths, but also a *conceptual* tool, to provide an explanation for the basic processes which underlie the physics of spin resonance strengths. This has been demonstrated with the successful theoretical explanation and fitting [7,8] of experimental measurements of spin–flip resonance strengths, for stored beams of polarized protons [9] and deuterons [10]. Although other formalisms can in principle be modified to calculate spin–flip resonance strengths, the spin response formalism was the only one actually in existence, with quantitative formulas, at the relevant time when the experimental data was published. Note that both [7,8] employed the spin response formalism. The calculation of the spin response functions has been coded into a publicly available program ASPIRRIN (Analysis of SPIn Resonances in RINgs).

The spin response formalism was also previously employed successfully for radiative polarization [11], to configure the ring optics at both the rings NIKHEF AmPS and MIT-Bates SHR to minimize the rate of depolarization and increase the stored polarization lifetime. Both rings were equipped with a single Siberian Snake (two solenoids in series) and circulated a beam of stored polarized electrons for use with internal gas jet targets. "Siberian Snakes" will be defined below.

More recently, a design for a new spin flipper at RHIC was published in Ref. [12], to operate with full strength Siberian Snakes. Shatunov and Mane [13] have employed the spin response formalism to analyze the design properties of a spin flipper in RHIC, and showed that the spin flipper parameters are highly sensitive to the details of the ring optics, including in particular the configuration of the spin rotators at the interaction points. Hence it is desirable to have an accurate formalism to calculate the spin response function for the RHIC optics.

A "Siberian Snake" is theoretically defined as any device which rotates the spin (on the reference orbit) through 180° around an axis in the accelerator's median plane, but acts overall as a drift space for the orbital motion. A partial Siberian Snake has a spin rotation angle of less than 180°. A spin rotator is defined as any device which rotates the spin from the vertical to the horizontal plane (or vice versa), but also acts overall as a drift space for the orbital motion. Mane [14] recently employed ASPIRRIN to analyze the available data on spin-flipping in rings with nearly full strength Siberian Snakes.

The structure of this paper is as follows. Section 2 describes our basic notation and terminology. Section 3 describes the spin response formalism, including the spin response functions. Section 4 describes the two principal theoretical formulas in the field, the Froissart–Stora formula [15] and the Derbenev–Kondratenko formula [16], and associated concepts of "adiabatic spin–flip", etc. Section 5 describes the application of the formalism to the calculation of various classes of resonance strengths. Section 6 describes a comparison with alternative formalisms and directions for future work. Section 7 concludes. Some ancillary material is relegated to an Appendix A.

2. Basic formalism

We treat a particle of charge e, mass m, with velocity $\boldsymbol{v}$ and spin $\boldsymbol{s}$. The Lorentz factor is $\gamma = 1/\sqrt{1-v^2/c^2}$. The magnetic moment anomaly will be denoted by $G = \frac{1}{2}(g-2)$. (For lepton rings it is more usual to write $a = \frac{1}{2}(g-2)$. We shall mainly write G in this paper.) The externally prescribed electric and magnetic fields of the accelerator will be denoted by $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. We employ cgs units, hence the Lorentz force is $e(\boldsymbol{E} + (\boldsymbol{v}/c) \times \boldsymbol{B})$. We denote the magnetic rigidity by $B\rho = p_0 c/e$, where p_0 is the reference momentum. We shall use the arc-length s along the reference orbit as the independent variable, and define the azimuth $\theta = s/R$, where the ring circumference is $2\pi R$. We employ a prime to denote differentiation with respect to θ, so $f' \equiv df/d\theta$. Our coordinate system is $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$, a right-handed orthogonal system where $\hat{\boldsymbol{x}}$ is radially outward, $\hat{\boldsymbol{y}}$ is along the reference orbit and $\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}}$ (this is vertical in a planar ring). The positive sense of circulation is counterclockwise around the ring. The horizontal and vertical betatron tunes will be denoted by ν_x and ν_z, respectively. We denote the orbital coordinates and momenta of a particle by $X^T = (x, p_x, z, p_z, \sigma, p_\sigma)$, where x and z are the horizontal and vertical coordinates of a particle, and p_x and p_z are their conjugate momenta, respectively. We scale all the orbital variables to be *dimensionless*, so what we call "x" would generally be denoted by x/R by other authors. Note also that x and z, etc., include *all* contributions to the motion (closed orbit imperfections, free and/or forced betatron oscillations, and dispersion and momentum offsets). The longitudinal motion is described by the dimensionless time lag (here ω_0 is the circulation frequency along the reference orbit) $\sigma = \theta - \omega_0 t$, and the relative momentum offset $p_\sigma = \Delta p/p_0$, where $\Delta p = p - p_0$. We also define the dimensionless scaled magnetic fields on the reference orbit $K_{x,y,z} = B_{x,y,z}/B_0$, where B_0 is a reference value given by the arc dipoles (i.e. excluding Siberian Snakes and spin rotators) $B_0 = (2\pi)^{-1} \int_{\text{arcs}} B_z \, d\theta$. The orbital equations of motion are given in Appendix A, in particular the relation between the canonical momenta p_x and p_z and the derivatives x' and z' is given in Eqs. (A.2a) and (A.2c). For motion in a planar ring, we introduce the "turning angle" Θ, defined via

$$\Theta = \int_0^\theta K_z(\tilde{\theta})\, d\tilde{\theta} = \int_0^\theta \frac{R}{\rho(\tilde{\theta})}\, d\tilde{\theta} \tag{2.1}$$

where ρ is the local bend radius in the horizontal plane. For nonplanar rings one requires the curvatures $\rho_x^{-1} = K_z/R$ and $\rho_z^{-1} = -K_x/R$. For future reference, we introduce here the periodic δ-function

$$\delta_p(\theta - \theta_0) = \sum_{j=-\infty}^{\infty} \delta(\theta - \theta_0 - 2j\pi). \tag{2.2}$$

We also define the fundamental skew-symmetric symplectic matrix

$$S = \begin{pmatrix} S_0 & & \\ & S_0 & \\ & & S_0 \end{pmatrix} \tag{2.3}$$

where S_0 is the 2×2 matrix

$$S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

Note that $S^2 = -1$ and $S^T = -S$.

The spin precession equation of motion in the externally prescribed electric and magnetic fields of the accelerator is known as the Thomas–BMT equation [17,18]. Using θ as the independent variable, the equation takes the form

$$\frac{d\boldsymbol{s}}{d\theta} = \boldsymbol{W} \times \boldsymbol{s}. \tag{2.5}$$

We subdivide the spin precession vector $\boldsymbol{W}$ into two parts $\boldsymbol{W} = \boldsymbol{W}_0 + \boldsymbol{w}$, where $\boldsymbol{W}_0$ denotes the precession vector on the reference orbit, and $\boldsymbol{w}$ denotes *all* of the other terms (contributions from closed orbit imperfection and free and forced orbital oscillations). We treat $\boldsymbol{w}$ as a perturbation. The spin tune is the ratio of the spin precession frequency to the orbit revolution frequency and is denoted by ν. It is well-known that in a planar ring the spin tune on the reference orbit is $\nu = \nu_0$, where $\nu_0 = G\gamma_0$ and γ_0 is the value of γ on the reference orbit. The components of $\boldsymbol{W}_0$ are

$$W_{0x} = \nu_0 K_x, W_{0y} = (1+G)K_y, W_{0z} = \nu_0 K_z. \tag{2.6}$$

If electric fields are neglected (basically, rf cavities and synchrotron oscillations), and the orbital and spin motion is due purely to magnetic fields, the terms in the spin precession vector can be expressed completely in terms of solenoidal fields and x and z and their derivatives (see Refs. [2,6]). The spin response formalism treats transverse coupling, and full or partial Siberian Snakes and spin rotators, but it is assumed that the orbital and spin motion is due purely to magnetic fields. Following Ref. [1], the components of $\boldsymbol{w}$ are given by

$$\begin{aligned} w_x &= (1+\nu_0)z'' + \left(\nu_0 + \frac{G}{\gamma_0}\right)K_x p_\sigma + (1+G)K_y x' \\ w_y &= (1+G)(K_x' x + K_z' z + \Delta K_y - K_y p_\sigma) - (\nu_0 - G)(K_x p_x + K_z p_z) \\ w_z &= -(1+\nu_0)x'' + \left(\nu_0 + \frac{G}{\gamma_0}\right)K_z p_\sigma + (1+G)K_y z'. \end{aligned} \tag{2.7}$$

We discuss briefly the neglect of electric fields. To treat bunched beams, we require rf cavities and synchrotron oscillations. In that case, the spin precession vector $\boldsymbol{W}$ cannot be expressed purely using (x, x', x'', z, z', z'') and solenoidal fields. Another feature of bunched beams is the existence of synchrotron sideband resonances. The consequences of sidebands are different for e^+e^- and hadron storage rings. In e^+e^- storage rings, the spacing between the sidebands is greater than the resonance strength $\nu_s \gg \varepsilon$, where ε denotes the spin flip resonance width and ν_s is the synchrotron tune. (Recall that the integrated field of a spin flipper, i.e. the value of ε, is under user control, so we can always satisfy the above inequality.) The range of the flipper frequency sweep must be such as to cross only one resonance (the parent resonance, because the sidebands are usually much weaker). In hadron rings, the opposite condition is true, $\varepsilon \gg \nu_s$ and the synchrotron oscillations are adiabatic to the spin precession. The range of the flipper frequency sweep in this case must be large enough to cross the spin tune spread of the beam. The main source of the spin tune spread is the beam energy spread. If the value of the synchrotron tune is small (as at RHIC), the spin response formalism will still give a good approximation for the spin response functions.

The next step is to solve Eq. (2.5) for motion on the reference orbit. The equation has one solution, denoted by $\boldsymbol{n}_0$, which is periodic around the ring, i.e.

$$\frac{d\boldsymbol{n}_0}{d\theta} = \boldsymbol{W}_0 \times \boldsymbol{n}_0 \tag{2.8a}$$

$$\boldsymbol{n}_0(\theta + 2\pi) = \boldsymbol{n}_0(\theta). \tag{2.8b}$$

There are also two other linearly independent solutions, say $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, such that $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{n}_0)$ is a right-handed orthonormal triad. However, in general $\boldsymbol{\eta}_{1,2}(\theta+2\pi) \neq \boldsymbol{\eta}_{1,2}(\theta)$. Instead, $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ precess around $\boldsymbol{n}_0$. Define the complex vector $\boldsymbol{\eta} = \boldsymbol{\eta}_1 - i\boldsymbol{\eta}_2$. Then $\boldsymbol{\eta}$ satisfies the properties

$$\frac{d\boldsymbol{\eta}}{d\theta} = \boldsymbol{W}_0 \times \boldsymbol{\eta} \tag{2.9a}$$

$$\boldsymbol{\eta}(\theta + 2\pi) = e^{i2\pi\nu}\boldsymbol{\eta}(\theta). \tag{2.9b}$$

The normalization is $|\boldsymbol{n}_0|^2 = 1$ and $|\boldsymbol{\eta}|^2 = 2$. For motion in a planar ring, the solutions for $\boldsymbol{n}_0$ and $\boldsymbol{\eta}$ are simply $\boldsymbol{n}_0 = \hat{\boldsymbol{z}}$ and $\boldsymbol{\eta} = (\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})e^{i\nu\Theta}$. Next we analyze the spin motion off the reference orbit. To do this we generalize $\boldsymbol{n}_0$ to a unit vector $\boldsymbol{n}$, which is a solution of Eq. (2.5) off-axis. The vector $\boldsymbol{n}$ has the property

$$\boldsymbol{n}(X, \theta + 2\pi) = \boldsymbol{n}(X, \theta). \tag{2.10}$$

We assume the motion is not on an orbital or spin resonance (and is not chaotic). If such a solution for $\boldsymbol{n}$ exists, it is unique (up to a $\pm$ sign). A spin resonance occurs when the spins precess coherently with the orbital oscillations and/or closed orbit imperfections. The general formula for the locations of the spin resonances is

$$\nu = m_0 + m_1\nu_{\rm I} + m_2\nu_{\rm II} + m_3\nu_{\rm III} \tag{2.11}$$

where m_0, m_1, m_2, and m_3 are integers, including zero, and $\nu_{\rm I}$, $\nu_{\rm II}$ and $\nu_{\rm III}$ are the tunes of the orbital eigenmodes. For spin flippers, the spin tune is held fixed at some value and the flipper tune is swept across the spin tune. The resonance condition is $\nu = m_0 \pm \nu_{rf}$ where ν_{rf} is the flipper tune. We parameterize $\boldsymbol{n}$ via

$$\boldsymbol{n} = (1 - |\zeta|^2)^{1/2}\boldsymbol{n}_0 + \Re(i\zeta\boldsymbol{\eta}^*). \tag{2.12}$$

The equation of motion for ζ is (from Eqs. (2.5), (2.8) and (2.9))

$$\frac{d\zeta}{d\theta} = (1 - |\zeta|^2)^{1/2}\boldsymbol{w}\cdot\boldsymbol{\eta} - i\boldsymbol{w}\cdot\boldsymbol{n}_0\zeta. \tag{2.13}$$

This is a nonlinear differential equation for ζ. We now make the approximation of retaining the orbital motion (oscillations and imperfection terms) to first order only. This yields the approximate equation

$$\frac{d\zeta}{d\theta} \simeq \boldsymbol{w}\cdot\boldsymbol{\eta}. \tag{2.14}$$

The approximation in Eq. (2.14) treats only resonances such that $|m_1| + |m_2| + |m_3|$ is 0 or 1 in Eq. (2.11). We refer to these as "first order resonances" below. In general, the most important higher order spin resonances (i.e. $|m_1| + |m_2| + |m_3| > 1$) are the synchrotron sideband resonances, which were briefly discussed above. The SLIM formalism by Chao [19] basically integrates Eq. (2.14) (see Section 6 for details), but the spin response formalism processes the right-hand side further, as will be explained in the next section.

3. Spin response formalism

We now derive the spin response functions. The formulation below mainly follows Ref. [1], but corrects some details in Refs. [1,4]. We first note that

$$\begin{aligned} \zeta' &= \boldsymbol{w}\cdot\boldsymbol{\eta} = (1+\nu_0)(z''\eta_x - x''\eta_z) + \cdots \\ &= (1+\nu_0)(z'\eta_x - x'\eta_z)' - (1+\nu_0)(z'\eta_x' - x'\eta_z') + \cdots \\ &\equiv (1+\nu_0)(z'\eta_x - x'\eta_z)' + C'. \end{aligned} \tag{3.1}$$

The solution is

$$\begin{aligned} \zeta &= (1+\nu_0)(z'\eta_x - x'\eta_z) + C \\ &= (1+\nu_0)[(p_z - \tfrac{1}{2}K_y x)\eta_x - (p_x + \tfrac{1}{2}K_y z)\eta_z] + C. \end{aligned} \tag{3.2}$$

The first term on the right-hand side is "already solved" (if we know the orbital motion) and is moreover nonresonant. We focus our attention on the function C. We solve the equation of motion for C using a Hamiltonian formalism, so we express the orbital motion in terms of dynamical variables (x, z, p_x, p_z). Then

$$\begin{aligned}C' = &-(1+\nu_0)[z'((1+G)K_y\eta_z - \nu_0 K_z\eta_y)\\ &- x'(\nu_0 K_x\eta_y - (1+G)K_y\eta_x)]\\ &+ [(1+G)(K_x' x + K_z' z + \Delta K_y - K_y p_\sigma)\\ &- (\nu_0 - G)(K_x p_x + K_z p_z)]\eta_y\\ &+ \left[\left(\nu_0 + \frac{G}{\gamma_0}\right)K_x p_\sigma + (1+G)K_y x'\right]\eta_x\\ &+ \left[\left(\nu_0 + \frac{G}{\gamma_0}\right)K_z p_\sigma + (1+G)K_y z'\right]\eta_z.\end{aligned} \tag{3.3}$$

The terms in x' and z' must be expressed using the canonical variables x, z, p_x and p_z. We express C' as a linear function of the orbital dynamical variables X, in the form

$$C' = w_0 + \mathscr{W}^T X. \tag{3.4}$$

Here $w_0 = (1+G)\Delta K_y\eta_y$. The solution for C is also a linear function of X, which we express in the form

$$C = f_0 + F^T S X. \tag{3.5}$$

From Eqs. (2.10), (2.12) and (3.2), it follows that C has the one-turn periodicity $C(X, \theta + 2\pi) = e^{i2\pi\nu}C(X, \theta)$. It then follows from Eq. (3.5) that

$$f_0(\theta + 2\pi) = e^{i2\pi\nu} f_0(\theta) \tag{3.6a}$$

$$F_j(\theta + 2\pi) = e^{i2\pi\nu} F_j(\theta). \tag{3.6b}$$

The equations of moton for F and X are given by a Hamiltonian formalism (with driving terms). The equations are

$$X' = SHX + Q \tag{3.7a}$$

$$F' = SHF + P. \tag{3.7b}$$

Here $Q^T = (0, -\Delta K_z, 0, \Delta K_x, 0, 0)$ is a vector of the driving terms of the closed-orbit imperfections, or also spin flippers, and $P = S\mathscr{W}$. The symmetric matrix H is the Hessian matrix of the orbital Hamiltonian $\mathscr{H}_{\text{orb}} : H_{ij} = \partial^2\mathscr{H}_{\text{orb}}/\partial X_i \partial X_j$. Since we treat only linear orbital dynamics, H does not depend on X. The function f_0 obeys a constraint-type equation, *after* the solution for F has been found:

$$f_0' = w_0 - F^T S Q = \Delta K_z F_1 - \Delta K_x F_3 + (1+G)\Delta K_y \eta_y. \tag{3.8}$$

This expression will be employed in Section 5 to derive expressions for resonance strengths. The components of the vector P are given by $P_6 = 0$ (because we have neglected electric fields) and

$$P_1 = (\nu_0^2 + G)K_x\eta_y - \nu_0(1+G)K_y\eta_x \tag{3.9a}$$

$$P_2 = -(1+G)K_x'\eta_y - \tfrac{1}{2}\nu_0(1+G)K_y^2\eta_z + \tfrac{1}{2}\nu_0(1+\nu_0)K_yK_z\eta_y \tag{3.9b}$$

$$P_3 = (\nu_0^2 + G)K_z\eta_y - \nu_0(1+G)K_y\eta_z \tag{3.9c}$$

$$P_4 = -(1+G)K_z'\eta_y + \tfrac{1}{2}\nu_0(1+G)K_y^2\eta_x - \tfrac{1}{2}\nu_0(1+\nu_0)K_yK_x\eta_y \tag{3.9d}$$

$$P_5 = \left(\nu_0 + \frac{G}{\gamma_0}\right)(K_x\eta_x + K_z\eta_z) - (1+G)K_y\eta_y. \tag{3.9e}$$

The above expressions correct some misprints in Ref. [1]. In general, the terms in K_yK_z and K_yK_x (for P_2 and P_4, respectively) do not exist, because most accelerators do not have magnets with both solenoidal and transverse fields simultaneously on the design orbit.

The functions F_1–F_5 are the "spin response" functions. (Note that $F_6 = -\partial C/\partial\sigma = 0$ because we have neglected synchrotron oscillations.) To calculate the spin response functions, we note that the same Hessian matrix H applies to both X and F, hence the transfer matrices for standard beamline elements can be used in the equation of motion for F (Eq. (3.7b)). The solution through any beamline (or element) can be expressed in map form as

$$X_{out} = MX_{in} + Z \tag{3.10a}$$

$$F_{out} = MF_{in} + Y \tag{3.10b}$$

where M is the (orbital) transfer matrix of the beamline. The expressions for Z are not required in this paper. Expressions for Y are given in Appendix A (and note that Y is nonzero only in beamline elements which have dipole or solenoidal fields on the reference orbit). Specifically, after one turn we obtain an expression of the form

$$F(\theta + 2\pi) = M_{\text{otm}}F(\theta) + Y_{\text{otm}} \tag{3.11}$$

where M_{otm} is the one-turn orbital map and Y_{otm} is obtained from the individual elements. Then, using Eq. (3.6b),

$$F(\theta) = (e^{i2\pi\nu} - M_{\text{otm}})^{-1} Y_{\text{otm}}. \tag{3.12}$$

This has a well-defined solution as long as $e^{i2\pi\nu}$ does not equal any of the eigenvalues of M_{otm}, i.e. the motion is not on a spin resonance.

The spin response functions F_1–F_5, as defined above, treat the orbital motion to only first order. With this caveat, and using Eq. (3.2), we obtain the first-order approximations for the change of $\boldsymbol{n}$ in response to momentum recoils (for example due to photon emission)

$$\begin{aligned}\frac{\partial\boldsymbol{n}}{\partial p_x} &\simeq \Re\left\{i\left[-(1+\nu_0)\eta_z + \frac{\partial C}{\partial p_x}\right]\boldsymbol{\eta}^*\right\}\\ &= \Re\{i[-(1+\nu_0)\eta_z + F_1]\boldsymbol{\eta}^*\}\end{aligned} \tag{3.13a}$$

$$\frac{\partial\boldsymbol{n}}{\partial p_z} \simeq \Re\left\{i\left[(1+\nu_0)\eta_x + \frac{\partial C}{\partial p_z}\right]\boldsymbol{\eta}^*\right\} = \Re\{i[(1+\nu_0)\eta_x + F_3]\boldsymbol{\eta}^*\} \tag{3.13b}$$

$$\gamma\frac{\partial\boldsymbol{n}}{\partial\gamma} \simeq \frac{\partial\boldsymbol{n}}{\partial p_\sigma} \simeq \Re\left\{i\frac{\partial C}{\partial p_\sigma}\boldsymbol{\eta}^*\right\} = \Re\{iF_5\boldsymbol{\eta}^*\}. \tag{3.13c}$$

In the last line we have made the approximation $\Delta E/E \simeq \Delta p/p$, which is valid for ultrarelativistic motion. The first two equations do not assume ultrarelativistic motion.

It is useful to express the spin response functions in terms of the orbital eigenmodes. This is helpful for formal comparison with other formalisms, or to gain a better insight into structure of the integrands for the functions. The most important example is the response to vertical momentum kicks, i.e. the function F_3. For uncoupled orbital motion, denote the vertical orbital mode by $f_z = \sqrt{\beta_z/R}e^{i\phi_z}$, where β_z is the vertical beta function and $d\phi_z/d\theta = R/\beta_z$. In this important case we obtain

$$\begin{aligned}F_3(\theta_0) = \frac{1}{2}\Bigg\{&\frac{f_z(\theta_0)}{e^{i2\pi(\nu-\nu_z)} - 1}\int_{\theta_0}^{\theta_0+2\pi}[(\nu_0^2 + G)K_z f_z^{*'} + (1+G)K_z' f_z^*]e^{i\nu_0\Theta}\,d\theta\\ &- \frac{f_z^*(\theta_0)}{e^{i2\pi(\nu+\nu_z)} - 1}\int_{\theta_0}^{\theta_0+2\pi}[(\nu_0^2 + G)K_z f_z' + (1+G)K_z' f_z]e^{i\nu_0\Theta}\,d\theta\Bigg\}.\end{aligned} \tag{3.14}$$

The above integral was first derived in Ref. [3], using a different notation, and was employed in Ref. [8] to fit experimental spin–flip measurements using a radial field rf dipole spin flipper.

Next we treat F_5. We first remark in passing that under the neglect of synchrotron oscillations, F_5 can be obtained from F_1 and

F_3 via (see Eq. (3.7b))

$$F_5' = K_z\left[\left(v_0+\frac{G}{\gamma_0}\right)\eta_z - F_1\right] + K_x\left[\left(v_0+\frac{G}{\gamma_0}\right)\eta_x + F_3\right] - (1+G)K_y\eta_y. \tag{3.15}$$

We next present one-turn integrals over lattice functions, to evaluate F_5. For uncoupled motion, the answer is similar to the so-called "Chao–Yokoya spin integrals" [20] (a more accessible reference is Ref. [21]), but the expressions below treat full coupling (but a synchrotron tune of zero). We return to Eq. (3.4) and express the orbital motion as a sum of (four-dimensional) orbital eigenmodes and a dispersion term. We write $\tilde{X}$ to denote a four-component column vector:

$$\tilde{X} \equiv \mathscr{F}A + \mathscr{D}p_\sigma. \tag{3.16}$$

The notation is as follows. First $\mathscr{D}$ denotes dispersion terms

$$\mathscr{D} = \begin{pmatrix} D_x \\ D_x' - \frac{1}{2}K_yD_z \\ D_z \\ D_z' + \frac{1}{2}K_yD_x \end{pmatrix} \tag{3.17}$$

where D_x and D_z are the horizontal and vertical dispersion functions. The A_j are complex amplitudes depending on initial conditions ($A_2 = A_1^*$ and $A_4 = A_3^*$) and $\mathscr{F}$ is a matrix composed of the orbital eigenmodes

$$\mathscr{F} = \begin{pmatrix} f_{1x} & f_{1x}^* & f_{2x} & f_{2x}^* \\ h_{1x} & h_{1x}^* & h_{2x} & h_{2x}^* \\ f_{1z} & f_{1z}^* & f_{2z} & f_{2z}^* \\ h_{1z} & h_{1z}^* & h_{2z} & h_{2z}^* \end{pmatrix}. \tag{3.18}$$

For example, for an uncoupled ring $f_{1x} = \sqrt{\beta_x/R}e^{i\phi_z}$ and $f_{1z} = 0$, where β_x is the horizontal beta function and $d\phi_x/d\theta = R/\beta_x$. In the vertical plane (also for an uncoupled ring) we encountered f_{2z} as f_z in Eq. (3.14). The normalization is (restricting S to a 4×4 matrix)

$$\mathscr{F}^\dagger S\mathscr{F} = \begin{pmatrix} 2i & & & \\ & -2i & & \\ & & 2i & \\ & & & -2i \end{pmatrix}. \tag{3.19}$$

We subdivide the function F_5 into two terms $F_5 = F_{5\gamma} + F_{5\beta}$. Here $F_{5\gamma}$ is the contribution from the direct dependence of $\boldsymbol{n}$ on the particle energy (i.e. p_σ) and $F_{5\beta}$ is due to the excitation of betatron oscillations due to the emission of quanta. Then from Eq. (3.4)

$$C' = \tilde{\mathscr{W}}^T(\mathscr{F}\mathscr{A} + \mathscr{D}p_\sigma) + \mathscr{W}_6 p_\sigma. \tag{3.20}$$

Here $\tilde{\mathscr{W}}$ denotes the first four components of $\mathscr{W}$. We have dropped the term in w_0 because it does not contribute to longitudinal momentum recoils. To calculate $F_{5\beta}$, note that under the assumption of a "point" photon emission, the total x and z etc., do not change and so (assuming ultrarelativistic motion)

$$\mathscr{F}\,\delta A = -\mathscr{D}\frac{\delta\gamma}{\gamma_0} \tag{3.21a}$$

$$\frac{\partial A}{\partial p_\sigma} = -\mathscr{F}^{-1}\mathscr{D}. \tag{3.21b}$$

Then solving for C and calculating $\partial C/\partial p_\sigma$ yields

$$F_{5\gamma}(\theta_0) = \frac{1}{e^{i2\pi v}-1}\int_{\theta_0}^{\theta_0+2\pi}(\tilde{\mathscr{W}}^T(\theta)\mathscr{D}(\theta) + \mathscr{W}_6(\theta))\,d\theta \tag{3.22a}$$

$$F_{5\beta}(\theta_0) = -\sum_{j=1}^{4}\frac{(\mathscr{F}^{-1}\mathscr{D})_j(\theta_0)}{e^{i2\pi(v+v_j)}-1}\int_{\theta_0}^{\theta_0+2\pi}(\tilde{\mathscr{W}}^T\mathscr{F})_j(\theta)\,d\theta. \tag{3.22b}$$

In the approximation of uncoupled motion, $F_{5\gamma}$ is the analog of the Chao-Yokoya spin integral for the synchrotron mode (in the limit of a synchrotron tune of zero), and $F_{5\beta}$ is the analog of the spin integrals over the betatron modes [20,21]. For ease of reference, we list the relevant components of W below (see Eq. (3.9), and we drop terms in K_xK_y and K_yK_z)

$$\mathscr{W}_1 = (1+G)K_x'\eta_y + \tfrac{1}{2}v_0(1+G)K_y^2\eta_z \tag{3.23a}$$

$$\mathscr{W}_2 = (v_0^2+G)K_x\eta_y - v_0(1+G)K_y\eta_x \tag{3.23b}$$

$$\mathscr{W}_3 = (1+G)K_z'\eta_y - \tfrac{1}{2}v_0(1+G)K_y^2\eta_x \tag{3.23c}$$

$$\mathscr{W}_4 = (v_0^2+G)K_z\eta_y - v_0(1+G)K_y\eta_z \tag{3.23d}$$

$$\mathscr{W}_6 = \left(v_0+\frac{G}{\gamma_0}\right)(K_x\eta_x + K_z\eta_z) - (1+G)K_y\eta_y. \tag{3.23e}$$

Recall that $\mathscr{W}_5 = -P_6 = 0$, and for ultrarelativistic motion it is usually acceptable to neglect the term in G/γ_0 in $\mathscr{W}_6$.

4. Polarization formulas

In this section we display the Froissart–Stora [15] and Derbenev–Kondratenko [16] formulas, and in the next section we present expressions to calculate the spin resonance strengths for use in the above formulas. For nonradiative polarization, Froissart and Stora [15] solved the spin precession equation of motion in a planar ring, for the passage across a single isolated resonance. The beam energy is varied at a uniform rate so that $G\gamma = v_{res} + \alpha\theta$, where v_{res} is the resonant spin tune, α is a constant, and the value of $G\gamma$ is increased from far below to far above the resonance. Froissart and Stora calculated the final asymptotic vertical polarization, for an initially vertically polarized beam, and derived what is now called the Froissart–Stora formula

$$\frac{P_f}{P_i} = 2e^{-\pi|\varepsilon|^2/(2|\alpha|)} - 1. \tag{4.1}$$

Here P_i and P_f are the initial and final vertical polarizations, respectively. The value of $|\varepsilon|$ is called the resonance strength. If $|\varepsilon|^2/|\alpha| \gg 1$, then $P_f/P_i \simeq -1$, i.e. the polarization direction reverses with negligible decrease of magnitude of the asymptotic polarization. This is called "adiabatic spin–flip". A spin flipper is operated so as to induce adiabatic spin–flip.

Using angle brackets to denote the secular component of a function, and treating only uncoupled vertical orbital motion, the resonance strength is given by [22]

$$\varepsilon = \langle [(G\gamma+1)z'' + i(G\gamma - G)K_zz' - i(1+G)K_z'z]e^{iG\gamma\Theta}\rangle. \tag{4.2}$$

We set $G\gamma = v_{res}$ in the above expression. The term in K_z' is essentially a fringe field term from the edges of the horizontal dipoles. The above integral coincides (up to differences of notation) with that given by Courant and Ruth [22]. (A version of Eq. (4.2) without fringe fields and approximating $G \ll 1$, which is valid for leptons, was also given in Ref. [2].) Courant and Ruth considered resonances where z was due to closed orbit imperfections ("imperfection" resonances) or free vertical betatron oscillations ("intrinsic" resonances), but they did not treat spin flippers. For a radial field rf dipole spin flipper, Eq. (4.2) is also applicable, where z denotes the forced vertical betatron oscillations induced by the spin flipper. For rings of more complicated topology, the spin response formalism gives a more general expression for the resonance strength. For a spin flipper, the spin tune is held fixed and the flipper tune is swept. Hence α in Eq. (4.1) is redefined as the rate of change of the spin flipper tune.

In a ring with full strength Siberian Snakes, the spin tune is $\nu = \frac{1}{2}$, so the condition of an isolated spin resonance is not satisfied. Hence the Snakes must be detuned slightly, to shift the spin tune away from $\frac{1}{2}$. Alternatively, an idea has been published in Ref. [12] to configure a new spin flipper at RHIC (comprising a pair of radial field rf dipoles) to cancel the driving term of one of the two spin resonances, so that the ring can be operated with full strength Snakes, at a spin tune of $\frac{1}{2}$. Note also that if the direction of $\boldsymbol{n}_0$ is not vertical, then the polarizations P_i and P_f in Eq. (4.1) are measured along the direction of $\boldsymbol{n}_0$. Also, Froissart and Stora performed their calculations only for a planar ring. Recently Mane [23] has published spin tracking studies which indicate that for rings with (nearly) full strength Siberian Snakes, the Froissart–Stora formula does not apply for spin flipping with radial or vertical field rf dipole spin flippers (but the formula is applicable for rf solenoid spin flippers).

Next, we treat radiative polarization. For an initially unpolarized electron or positron beam, the emission of spin–flip synchrotron radiation causes the polarization to build up spontaneously to an equilibrium level P_{eq} according to

$$P(t) = P_{eq}(1 - e^{-t/\tau_{pol}}). \tag{4.3}$$

This spontaneous polarizing effect was first calculated by Sokolov and Ternov [24], who obtained an equilibrium value, for motion in a uniform vertical magnetic field (the "Sokolov–Ternov polarization"), of $P_{ST} = 8/(5\sqrt{3}) \simeq 0.924$. Derbenev and Kondratenko [16] calculated the formulas for the equilibrium polarization P_{eq} and the polarization buildup time τ_{pol} in high energy e^+e^- storage rings, including the effects of spin–orbit coupling and spin resonances:

$$P_{eq} = -\frac{8}{5\sqrt{3}}\frac{\alpha_-}{\alpha_+} \tag{4.4a}$$

$$\tau_{pol}^{-1} = \frac{5\sqrt{3}}{8}\frac{e^2\hbar\gamma^5}{m^2c^2}\alpha_+. \tag{4.4b}$$

Eq. (4.4a) is known as the Derbenev–Kondratenko formula. Here

$$\alpha_- = \left\langle \oint \frac{d\theta}{|\rho|^3}\hat{\boldsymbol{b}}\cdot(\boldsymbol{n}-\boldsymbol{d})\right\rangle \tag{4.5a}$$

$$\alpha_+ = \left\langle \oint \frac{d\theta}{|\rho|^3}\left[1 - \frac{2}{9}(\boldsymbol{n}\cdot\hat{\boldsymbol{v}})^2 + \frac{11}{18}|\boldsymbol{d}|^2\right]\right\rangle. \tag{4.5b}$$

The integral over θ is around the circumference of the ring, and the angle brackets denote an equilibrium average over the orbital phase space. Here $\hat{\boldsymbol{v}}$ is a unit vector parallel to the local velocity vector (tangent to the particle orbit), $\dot{\boldsymbol{v}}$ is the time derivative of $\boldsymbol{v}$, with $\hat{\dot{\boldsymbol{v}}} = \dot{\boldsymbol{v}}/|\dot{\boldsymbol{v}}|$ (normal to the particle orbit) and $\hat{\boldsymbol{b}} = \boldsymbol{v}\times\dot{\boldsymbol{v}}/|\boldsymbol{v}\times\dot{\boldsymbol{v}}|$ is the binormal vector, and ρ is either ρ_x or ρ_z, the local radius of curvature of the orbit. For motion in a locally uniform magnetic field, $\hat{\boldsymbol{b}}$ is a unit vector in the direction of the local magnetic field, hence the notation. Note that in Eq. (4.5), $\boldsymbol{n}$ is not restricted to a first order approximation; it includes the orbital motion (including imperfection terms) to *all* orders. The vector $\boldsymbol{d}$ is the partial derivative with respect to the relative energy offset $\Delta E/E$ of a particle

$$\boldsymbol{d} = \gamma\frac{\partial\boldsymbol{n}}{\partial\gamma}. \tag{4.6}$$

The vector $\gamma(\partial\boldsymbol{n}/\partial\gamma)$ is known as the "spin–orbit coupling vector". The spin response formalism gives a first-order approximation for $\gamma(\partial\boldsymbol{n}/\partial\gamma)$, via the function F_5 (see Eq. (3.13c)). The contributions of vertical momentum recoils to the equilibrium polarization were studied in Ref. [25], and also in Ref. [26]. The latter authors derived a formula involving $\partial\boldsymbol{n}/\partial p_z$, i.e. the function F_3 (see also Ref. [27]).

5. Resonance strengths

5.1. Nonradiative polarization

The resonance strength for use in the Froissart–Stora formula Eq. (4.1) is obtained from Eq. (3.8). If the resonant tune is κ, we set $\nu = \kappa$ and evaluate the integral [1]

$$\varepsilon = \frac{1}{2\pi}\left|\oint[\Delta K_z F_1 - \Delta K_x F_3 + (1+G)\Delta K_y\eta_y]_{\nu=\kappa}\,d\theta\right|. \tag{5.1}$$

The above integral applies to imperfection resonances, where ΔK_x and ΔK_z arise from closed orbit imperfections. (Typically ΔK_z contributes only in rings with coupling or partial Snakes.) One can also consider ΔK_y from a solenoid partial Snake as the driving term of a resonance. The above integral also applies for perturbing fields $\Delta K_{x,y,z}$ from spin flippers. The resonance strength formula in Eq. (4.2) applies only for an uncoupled planar ring. The expression in Eq. (5.1) is applicable to rings with arbitrary structure. For intrinsic resonances, the answer is, for the nth orbital eigenmode $\mathscr{F}_n$ (the nth column of the matrix in Eq. (3.18), see also Eq. (3.20) and ignore the terms in p_σ)

$$\varepsilon_n = \left|\frac{A_n}{2\pi}\int_{\theta_0}^{\theta_0+2\pi}[\tilde{\mathscr{W}}^T\mathscr{F}_n(\theta)]_{\nu=k-\nu_n}\,d\theta\right|. \tag{5.2}$$

The resonant tune is $k - \nu_n$, where k is an integer. The amplitude of A_n can be set to the normalized emittance, for example. Typically, in planar uncoupled rings only the vertical oscillation mode is significant, but the above expression treats also linear coupled motion.

We present some important examples for spin flippers. Let the integrated field of an rf solenoid spin flipper be

$$\int B_{\rm rfs}\,d\ell = B_{\rm osc}L\cos(\nu_{rf}\theta + \chi). \tag{5.3}$$

Here $B_{\rm osc}L$ is the peak integrated field and ν_{rf} is the rf solenoid tune and χ is an initial phase. A spin resonance occurs when $\nu = \kappa_{\rm rfs} = k \pm \nu_{rf}$. We require that the value of $\kappa_{\rm rfs}$ is not an integer or half-integer, otherwise two spin resonances are crossed simultaneously and we do not satisfy the requirements for an adiabatic spin flip. Then

$$\Delta K_y = \frac{1}{2}\frac{B_{\rm osc}L}{B\rho}(e^{i(\nu_{rf}\theta+\chi)} + e^{-i(\nu_{rf}\theta+\chi)})\delta_p(\theta - \theta_{\rm rfs}). \tag{5.4}$$

By hypothesis only one of the two terms is resonant. The resonance strength is

$$\varepsilon_{\rm rfs} = \frac{1}{2\pi}\left|\oint(1+G)\Delta K_y\eta_y\,d\tilde{\theta}\right|_{\nu=\kappa_{\rm rfs}} = \frac{1}{4\pi}\left|(1+G)\frac{B_{\rm osc}L}{B\rho}\eta_y(\theta=\theta_{\rm rfs})\right|. \tag{5.5}$$

For a planar uncoupled ring, $\eta_y = -ie^{iG\gamma\Theta}$ and we obtain the well known result

$$\varepsilon_{\rm rfs} = \frac{1}{4\pi}\left|(1+G)\frac{B_{\rm osc}L}{B\rho}\right|. \tag{5.6}$$

Next we treat an rf dipole. By a similar derivation to the above, and reusing some of the same notation (which should not cause ambiguity), let the integrated field of an rf dipole spin flipper be

$$\int B_{\rm rfd}\, d\ell = B_{\rm osc} L \cos(\nu_{rf}\theta + \chi). \tag{5.7}$$

A spin resonance occurs when $\nu = \kappa_{\rm rfd} = k \pm \nu_{rf}$. We also require that the value of $\kappa_{\rm rfd}$ not be an integer or half-integer. Then for a radial field rf dipole spin flipper (say "rrfd")

$$\Delta K_x = \frac{1}{2}\frac{B_{\rm osc}L}{B\rho}(e^{i(\nu_{rf}\theta+\chi)} + e^{-i(\nu_{rf}\theta+\chi)})\delta_p(\theta - \theta_{\rm rrfd}). \tag{5.8}$$

Again by hypothesis only one of the two terms is resonant, so the resonance strength is

$$\varepsilon_{\rm rrfd} = \frac{1}{4\pi}\frac{B_{\rm osc}L}{B\rho}|F_3(\theta_{\rm rrfd})|. \tag{5.9}$$

A similar derivation yields the resonance strength for a vertical field rf dipole spin flipper (say "vrfd")

$$\varepsilon_{\rm vrfd} = \frac{1}{4\pi}\frac{B_{\rm osc}L}{B\rho}|F_1(\theta_{\rm vrfd})|. \tag{5.10}$$

If there are multiple rf dipoles in the ring, we must sum the individual contributions taking into account the relative amplitudes and phases of the individual rf dipoles.

The spin response function F_3, in conjunction with Eq. (5.9), was employed in Ref. [7] to fit some recent experimental measurements of spin flip resonance widths for stored beams of polarized protons [9] and deuterons [10] at the COSY storage ring. The value of F_3 was calculated using ASPIRRIN. The deuteron spin–flip data were also fitted in Ref. [8], by evaluating the integral in Eq. (5.9) (but not using the ASPIRRIN program). In addition, Mane [14] analyzed the available data on spin-flipping in rings with nearly full strength Siberian Snakes. In all the cases where an rf dipole was used to flip the spins, it was a *vertical* field rf dipole, and so the relevant spin response function was F_1, also calculated using ASPIRRIN.

Shatunov and Mane [13] have used the spin response formalism to analyze a spin flipper consisting of two radial field rf dipoles at RHIC, for use with full strength Siberian Snakes and a spin tune of $\frac{1}{2}$. It was shown, for example, that the value of F_3 in RHIC depends sensitively on the configuration of the spin rotators at the STAR and PHENIX interaction points.

5.2. Radiative polarization

As noted above, for radiative polarization, the Derbenev–Kondratenko formula requires the function $\gamma(\partial \boldsymbol{n}/\partial\gamma)$ (see Eq. (4.5)), and the first order approximation is given by Eq. (3.13c), i.e. $\Re\{iF_5\boldsymbol{\eta}^*\}$. Hence we need to calculate F_5. However, the use of the spin response formalism has some limitations in this regard. The spin response formalism, as described in this paper and implemented in ASPIRRIN, calculates F_5 only for an ideal lattice. However, for most planar storage rings, $\gamma(\partial \boldsymbol{n}/\partial\gamma)$ *vanishes* in an ideal ring, and the nonzero value of $\gamma(\partial \boldsymbol{n}/\partial\gamma)$ arises from driving terms from the closed orbit imperfections. To calculate $\gamma(\partial \boldsymbol{n}/\partial\gamma)$, it is therefore necessary to reference the "unperturbed" orbital and spin motion to the imperfect closed orbit. The matter will be discussed in more detail in Section 6, in connection with the SLIM formalism [19], which is an alternative formalism to calculate first-order spin resonances in storage rings.

Nevertheless, the function F_5 *has* been employed successfully to aid in the design of storage rings, with positive results. We briefly describe the work in Ref. [11], to minimize the value of $|\gamma(\partial \boldsymbol{n}/\partial\gamma)|$ in rings with a single solenoid Snake. At both NIKHEF AmPS and MIT-Bates SHR, the "Snake system" consisted of two superconducting solenoids in series, with ancillary normal and skew quadrupoles to compensate the betatron coupling. Also in both cases, the Snake systems were designed and built at the Budker Institute for Nuclear Physics, Novosibirsk, by a team led by one of the authors (Shatunov). At both rings, a beam from a polarized electron source was injected into the ring. However, if the value of $|\gamma(\partial \boldsymbol{n}/\partial\gamma)|$ is too large, the spin–flip synchrotron radiation might cause unacceptably rapid depolarization, reducing the stored polarization lifetime. Hence one aims to minimize the value of $|\gamma(\partial \boldsymbol{n}/\partial\gamma)|$.

Let us say that the Snake system (solenoid and quadrupoles) occupies the interval $0<\theta<\theta_1$ and the rest of the ring occupies the interval $\theta_1<\theta<2\pi$. We make the reasonable assumption that there is no dispersion in the insertion region ($D_x = 0, D_z = 0$). We also assume that the transverse coupling introduced by the solenoids is fully compensated by the quadrupoles of the Snake system. We refer the reader to Ref. [11] for technical details. Recall that we can subdivide the function F_5 into two terms $F_5 = F_{5\gamma} + F_{5\beta}$. For a ring with a single solenoid Snake, the solution for $F_{5\gamma}$ outside the Snake system is (using Eqs. (3.22a) and (3.22b))

$$F_{5\gamma} = \frac{\pi}{2}\sin(\pi\nu_0) + i\nu_0(\pi - \Theta). \tag{5.11}$$

For a system of two solenoids in series, the function $F_{5\beta}$ is given by

$$F_{5\beta} = -\frac{\nu_0\pi}{4\cos(\pi\nu_x)}[\cos(\pi\nu_0)\Im(e^{i\pi\nu_x}J(\theta)G_{Ix}^*) + i\Im(e^{i\pi\nu_x}J(\theta)G_{Iz}^*)], \tag{5.12}$$

where $J(\theta) = f_{Ix}D_x' - f_{Ix}'D_x$ and $G_{Ix,z} = f_{Ix,z(out)}' - f_{Ix,z(in)}'$. Here f_I (see Eq. (3.18)) denotes the horizontal-like Floquet mode (a horizontal betatron oscillation in an uncoupled ring). The derivatives $f_{Ix,z(in)}'$ and $f_{Ix,z(out)}'$ are taken at the entrance of the first solenoid and the exit of the second solenoid, respectively. We consider here a solenoid with a thin edge. The first solenoid entrance is the point just after the first edge and the second solenoid exit is the point just before its second edge. To obtain Eq. (5.12) we approximated $g-2=0$, but this is an adequate approximation for leptons.

It is impossible to make F_5 vanish completely because for this model of a ring, $F_{5\gamma}$ does not depend on the machine optics. By imposing the constraints $f_{Ix(out)}' = f_{Ix(in)}'$ and $f_{Iz(out)}' = f_{Iz(in)}'$ on the lattice design, one can set $G_{Ix,z} = 0$, which leads to $F_{5\beta} = 0$. This was done at both AmPS and SHR and successfully decreased the magnitude of $|\gamma(\partial \boldsymbol{n}/\partial\gamma)|$, and increased the stored polarization lifetime [11].

6. Discussion of other formalisms and future work

We comment briefly on other formalisms to treat first-order spin resonances. The formalism by Courant and Ruth [22] calculates the strengths of imperfection and intrinsic resonances (and can be extended, in principle, to treat spin flippers), but is restricted to planar uncoupled rings and motion in magnetic fields only (no synchrotron oscillations). We discuss in more detail the "SLIM formalism" by Chao [19]. In the SLIM formalism, the unperturbed orbital motion and the spin basis vectors $\boldsymbol{n}_0$ and $\boldsymbol{\eta}$ are defined with respect to the imperfect closed orbit. The SLIM formalism treats only linear orbital dynamics, and the orbital motion is expanded to the first order in the orbital amplitudes. The perturbations to the spin motion (i.e. the off-axis vector $\boldsymbol{n}$) are calculated to the first order in the orbital amplitudes. Synchrotron oscillations are included. Basically, we express Eq. (2.14) in the form

$$\zeta' = \mathscr{U}^T X. \tag{6.1}$$

There is no inhomogenous term as in Eq. (3.4) because the motion is with respect to the imperfect closed orbit. We expand the orbital motion in a sum of (six-dimensional) eigenmodes

$$X = \sum_{j=\pm1,\pm2,\pm3} \mathscr{B}_j \mathscr{E}_j \tag{6.2}$$

where $\mathscr{B}_{-j} = \mathscr{B}_j^*$ specify the initial conditions and $\mathscr{E}_{-j} = \mathscr{E}_j^*$ are the eigenmodes. The normalization is similar to Eq. (3.19): $\mathscr{E}_j^\dagger S \mathscr{E}_j = 2i$ for $j>0$ and the complex conjugate for $j<0$. The solution for ζ is

$$\zeta(\theta_0) = \sum_{j=\pm1,\pm2,\pm3} \frac{\mathscr{B}_j}{e^{i2\pi(\nu+\nu_j)} - 1} \int_{\theta_0}^{\theta_0+2\pi} \mathscr{U}^T \mathscr{E}_j \, d\theta. \tag{6.3}$$

Here $\nu_{-j} = -\nu_j$. The expression on the right-hand side can be expressed as a matrix operator acting on $\mathscr{E}_j(\theta_0)$, using so-called "generalized matrices", and the solution can be used to derive expressions for $\partial \boldsymbol{n}/\partial p_x$, $\partial \boldsymbol{n}/\partial p_z$ and $\gamma \partial \boldsymbol{n}/\partial \gamma \simeq \partial \boldsymbol{n}/\partial p_\sigma$ (see Ref. [19] for details). For example, Barber and Mane [26] employed the SLIM formalism to calculate $\partial \boldsymbol{n}/\partial p_z$ to obtain the contribution of the vertical momentum recoils to the equilibrium radiative polarization in a storage ring. Additionally, Mane [27] established the connection of the Barber-Mane derivation to the spin response function F_3.

In the treatment presented in Ref. [19], only free orbital oscillations are treated, not driven (coherent) oscillations. Hence spin flippers are not handled (however Barber [28] has recently extended the SLIM formalism to do so). Also the SLIM formalism does not isolate the nonresonant terms as in Eq. (3.2), and imperfection and intrinsic resonances are not treated. However, the SLIM formalism treats the orbital motion using a fully coupled six-dimensional symplectic formalism, hence rf cavities, synchrotron oscillations and bunched beams, and also synchro-betatron coupling are supported (as well as transverse betatron coupling). This is significant for the calculation of the (first-order) approximation for the vector $\boldsymbol{d} = \gamma(\partial \boldsymbol{n}/\partial \gamma)$. It is well-known that in most planar storage rings (without Siberian Snakes or spin rotators), the value of $\gamma(\partial \boldsymbol{n}/\partial \gamma)$ vanishes in an ideal ring, and the actual nonzero value of $\gamma(\partial \boldsymbol{n}/\partial \gamma)$ is due to the closed orbit imperfections. The SLIM formalism [19] has been widely used in this context to calculate (first-order) $\gamma(\partial \boldsymbol{n}/\partial \gamma)$ for many storage rings. However, as we have stated earlier in this paper, ASPIRRIN does not calculate the contribution of closed orbit imperfections to the spin response function F_5 (hence $\gamma(\partial \boldsymbol{n}/\partial \gamma)$). Note that both SLIM and ASPIRRIN treat rings of arbitrary structure, e.g. Siberian Snakes and spin rotators. For future work on the spin response formalism (and the ASPIRRIN program), one topic is to include the effects of bunched beams. Another topic is to modify the formalism for the calculation of F_5, to include also contributions from lattice imperfections.

7. Conclusion

As stated in the introduction, the major reference on the spin response formalism [1] is in Russian. The present document is intended to provide a description of the principal details of the spin response formalism in an English-language paper, to make the spin response formalism more generally accessible to workers in the field. The spin response formalism has recently demonstrated its utility by the successful theoretical explanation and quantitative fitting [7,8] of experimental measurements of spin flip resonance widths for stored beams of polarized protons [9] and deuterons [10] at the COSY storage ring.

Additionally, in a modern development of current interest, Shatunov and Mane [13] have used the spin response formalism to analyze a spin flipper consisting of two radial field rf dipoles at RHIC, for use with full strength Siberian Snakes and a spin tune of $\frac{1}{2}$. The value of the spin response function F_3 (calculated using ASPIRRIN) indicates that the spin flipper parameters must be set with careful attention to the ring optics, including in particular the configuration of the spin rotators at the interaction points.

Also in the context of rings with Snakes, but this time for radiative polarization with a single (solenoid) Snake in the ring, Ptitsyn and Shatunov [11] employed the spin response function F_5 and derived design criteria to minimize the value of $|\gamma(\partial \boldsymbol{n}/\partial \gamma)|$. These criteria for the ring optics were successfully implemented at the storage rings AmPS (NIKHEF) and SHR (MIT-Bates) to decrease the depolarization rate caused by the spin–flip synchrotron radiation, and thereby to increase the stored polarization lifetime.

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Appendix A

We present expressions for the vector Y, which is the inhomogenous term in the map equation for F (see Eq. (3.10b)) for horizontal bends with combined-focusing quadrupole gradients (HBEND), and similarly vertical bends (VBEND) and solenoids without fringe fields (SOLENOID). First we list some definitions and state the equations of motion

$$g_{\rm quad} = \frac{R}{B_0} \frac{\partial B_z}{\partial x} \tag{A.1a}$$

$$g_x = K_z^2 + g_{\rm quad} \tag{A.1b}$$

$$g_z = K_x^2 - g_{\rm quad} \tag{A.1c}$$

$$\kappa_c = \frac{R}{B_0}\left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right). \tag{A.1d}$$

Here $g_{\rm quad}$ and κ_c describe quadrupole focusing and linear coupling, respectively. The orbital equations of motion are

$$x' = p_x + \tfrac{1}{2}K_y z \tag{A.2a}$$

$$p_x' = -g_x x - \kappa_c z + K_z p_\sigma + \tfrac{1}{2} K_y (p_z - \tfrac{1}{2}K_y x) - \Delta K_z \tag{A.2b}$$

$$z' = p_z - \tfrac{1}{2}K_y x \tag{A.2c}$$

$$p_z' = -g_z z - \kappa_c x - K_x p_\sigma - \tfrac{1}{2} K_y (p_x + \tfrac{1}{2}K_y z) + \Delta K_x \tag{A.2d}$$

$$\sigma' = \frac{1}{\gamma_0^2} p_\sigma - (K_x z - K_z x) \tag{A.2e}$$

$$p_\sigma' = \frac{1}{\beta_0^2} \frac{eVh\cos\phi_0}{E_0} \sigma. \tag{A.2f}$$

Here V is the voltage in the rf cavities, ϕ_0 is the synchronous rf phase, h is the harmonic number and E_0 is the reference particle energy (with obvious meanings for β_0 and γ_0). In the spin response formalism, we neglect the synchrotron oscillations. We derive the Hessian matrix H from the above equations.

We now present expressions for the vector Y. For a horizontal bend $K_x = K_y = 0$, and the nonvanishing P_j are

$$P_3 = (\nu_0^2 + G) K_z \eta_y \tag{A.3a}$$

$$P_4 = -(1+G) K_z' \eta_y \tag{A.3b}$$

$$P_5 = \left(v_0 + \frac{G}{\gamma_0}\right) K_z \eta_z. \tag{A.3c}$$

The solution for the nonvanishing Y_j is (with $g_z = -g$ since $K_x = 0$)

$$Y_3 = \frac{(v_0^2 + G)K_z}{(v_0 K_z)^2 + g}[v_0 K_z (M_{33}\eta_{x0} - \eta_x) + M_{43}\eta_{y0}] - (1+G)K_z M_{34}\eta_{y0} \tag{A.4a}$$

$$Y_4 = \frac{(v_0^2 + G)K_z g}{(v_0 K_z)^2 + g}[M_{33}\eta_{y0} - \eta_y + v_0 K_z M_{34}\eta_{x0}] + (1+G)K_z(\eta_y - M_{33}\eta_{y0}) \tag{A.4b}$$

$$Y_5 = \left(v_0 + \frac{G}{\gamma_0}\right) K_z \eta_z \theta. \tag{A.4c}$$

The solution for a vertical bend can be written by analogy, with $g_x = g$ since $K_z = 0$. The nonvanishing Y_j are

$$Y_1 = \frac{(v_0^2 + G)K_x}{(v_0 K_x)^2 - g}[-v_0 K_x (M_{11}\eta_{z0} - \eta_z) + M_{21}\eta_{y0}] - (1+G)K_x M_{12}\eta_{y0} \tag{A.5a}$$

$$Y_2 = \frac{(v_0^2 + G)K_x g}{(v_0 K_x)^2 - g}[-(M_{11}\eta_{y0} - \eta_y) + v_0 K_x M_{12}\eta_{z0}] + (1+G)K_x(\eta_y - M_{11}\eta_{y0}) \tag{A.5b}$$

$$Y_5 = \left(v_0 + \frac{G}{\gamma_0}\right) K_x \eta_x \theta. \tag{A.5c}$$

We neglect the entrance and exit fields of a solenoid. Then $K_x = K_z = 0$ and

$$P_1 = -v_0(1+G)K_y\eta_x \tag{A.6a}$$

$$P_2 = -\tfrac{1}{2}v_0(1+G)K_y^2\eta_z \tag{A.6b}$$

$$P_3 = -v_0(1+G)K_y\eta_z \tag{A.6c}$$

$$P_4 = \tfrac{1}{2}v_0(1+G)K_y^2\eta_x \tag{A.6d}$$

$$P_5 = -(1+G)K_y\eta_y \tag{A.6e}$$

and $P_6 = 0$. For $G \neq 0$, the solution is

$$Y_1 = -v_0\frac{1+G}{G}[-\eta_z(1-\cos(GK_y\theta)) + \eta_x\sin(GK_y\theta)] \tag{A.7a}$$

$$Y_2 = -v_0\frac{1+G}{G}(\tfrac{1}{2}K_y)[\eta_x(1-\cos(GK_y\theta)) + \eta_z\sin(GK_y\theta)] \tag{A.7b}$$

$$Y_3 = -v_0\frac{1+G}{G}[\eta_x(1-\cos(GK_y\theta)) + \eta_z\sin(GK_y\theta)] \tag{A.7c}$$

$$Y_4 = -v_0\frac{1+G}{G}(\tfrac{1}{2}K_y)[\eta_z(1-\cos(GK_y\theta)) - \eta_x\sin(GK_y\theta)] \tag{A.7d}$$

$$Y_5 = -(1+G)K_y\eta_y\theta \tag{A.7e}$$

and $Y_6 = 0$. For $|G| \ll 1$ (or $|GK_y\theta| \ll 1$), this reduces to $Y \simeq P\theta$, which is the solution given in Ref. [1].

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