Stern-Gerlach Electron Deflection Beamline Design

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1 Permanent magnet quadrupole lenses

This note describes a beamline design for detection and measurement of the Stern-Gerlach (SG) deflection of a suitably-polarized relativistic electric beam at CEBAF. The SG deflection occurs in the quadrupoles shown in Figure 1.

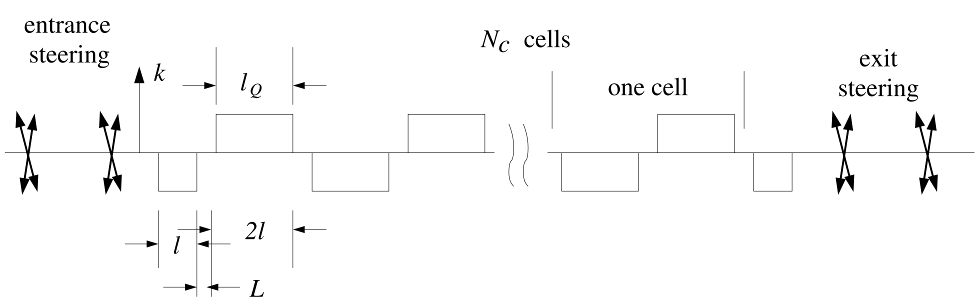

Figure 1: Stern-Gerlach deflection occurs in the FODO section of quadrupoles shown. For this note, numerical calculations assume quadrupoles of length $l_Q = 2l = 0.02\,\mathrm{m}$ separated by $L = 0.005\,\mathrm{m}$. Entrance and exit steering to allow for quadrupole strength and positioning errors is adjusted to cancel an overall steering error. Quadrupole strength coefficient k is defined in the text. For a proposed test at CEBAF there would be $N_c = 4$ FODO cells.

The electron beam is conveyed through a beamline consisting of N_c identical cells, each consisting of one focusing and one defocusing permanent magnet quadrupole of identical design, with length l_Q and constant magnetic field gradient $\partial B_x/\partial x$. Spin-dependent SG deflections occur in each of the quadrupoles, and the betatron phase advances are arranged so that the deflections all add constructively. The total SG angular deflection is therefore greater than each individual factor by a factor $2N_c$. For an initial test described in this note I choose $N_c = 4$ but, for an eventual apparatus, N_c could, be at least several times greater, depending on tolerance issues to be discussed.

Since the design uses permanent magnets, any realization of the design is static, specific to a particular electron beam energy. But the design scales easly to other energies and choice of N_c. The assumed quadrupoles are patterned after permanent magnet quadrupole described in Table III of a paper by Li and Musumeci[1] and in reference[2].

During its brief passage through a short quadrupole, the orbit of a momentum p electron, offset transversely by displacement x, can be taken to be circular with radius r such that its angular deflection is $\theta = l_Q/r$. The bend radius is determined by the centripetal force equation,

$$\frac{pv}{r} = evB = ev\frac{\partial B_x}{\partial x}\,x \tag{1}$$

Treating the quadrupole as thin, and re-arranging this equation, the integrated particle deflection angle during passage is

$$\theta = \frac{l_Q}{r} = \frac{cl_Q\langle\partial B_x/\partial x\rangle x}{pc/e}, \tag{2}$$

where the particle momentum has been incorporated as pc/e for the convenience of evaluating pc/e in the MKS unit of voltage. Ascribing this deflection to a quadrupole of strength (i.e. inverse focal length) $q = 1/f$, the deflection angle is $\pm qx$ where

$$q = \pm\frac{\theta}{x} = \pm\frac{c}{pc/e}\Big[l_Q\langle\partial B_x/\partial x\rangle\Big] = \pm\mathcal{C}_\gamma(3\times 10^8)/(0.511\times 10^6)\Big[\frac{l_Q\langle\partial B_x/\partial x\rangle}{\gamma_e}\Big]. \tag{3}$$

Here the electrons, with rest energy 0.511 MeV, have been assumed to be fully relativistic; i.e. $pc = \mathcal{E}_e = \gamma_e m_e c^2$. But a coefficient $\mathcal{C}_\gamma \approx 1$ has been included to allow for not quite fully relativistic beams. The lowest practical CEBAF particle kinectic energy is about 0.5 Mev, which is not fully relativistic—rather $\gamma_e \approx 2$ and $v/c \approx 0.87$. We make only a 13 percent error, worst case, by simply seting $\mathcal{C}_\gamma = 1$, to produce

$$q \approx \pm 587\mathrm{T}^{-1}\mathrm{m}^{-1}\Big[\frac{l_Q\langle\partial B_x/\partial x\rangle}{\gamma_e}\Big]. \tag{4}$$

The γ_e factor inside the square bracket "cancels" the momentum dependence, allowing the lens strength to be expressed as an inverse focal length. To the extent the field gradient can be increased without limit, the lens can be treated as purely geometric (i.e. independent of momentum) simply by increasing the $l_Q\partial B_x/\partial x$ proportional to γ_e. Though increasing l_Q also has the effect of increasing the lens inverse focal length, this (adversely) alters the optics.

With the square bracket expressing the integrated quadrupole strength in Tesla, this formula produces inverse focal length q in inverse meter units. For numerical examples in this note I take $l_q = 0.02\,\mathrm{m}$ and (already achievable) field gradient $\partial B_x/\partial x = 500\,\mathrm{T/m}$ as nominal values. Higher field gradient, $\partial B_x/\partial x = 1000\,\mathrm{T/m}$, at shorter length, $l_Q = 0.01\,\mathrm{m}$ is expected[3] to be achievable. This would yield the same length-strength product of 10 T, but be more useful in the (important) sense of allowing a lens of the same strength to be shorter relative to its focal length.

Limited only by the maximum achievable permanent magnetic field gradient, even with careful element alignment and coherent multiplication of the displacement by the number of quadrupoles in the beamline, the Stern-Gerlach deflection can be expected to be only comparable in magnitude with deflection caused by misaligned quadrupoles. As explained in previous reports, this spurious excitation will be suppressed by the interleaving of opposite-polarization A and B beams. This has the effect of shifting the spectral frequency of the SG deflection to one half the spectral frequency of the spurious deflection, allowing the contributions to be separated in a frequency-sensitive BPM.

2 Beamline optics

Currently the upstream beam phase space ellipse is not well known. I tentatively assume, for both planes,

$$\begin{aligned}
\epsilon &= \frac{1.0 \times 10^{-6}\,\mathrm{m}}{\gamma_e}, \\
\sigma^0 &= \frac{50\,\mu\mathrm{m}}{\sqrt{\gamma_e}}, \\
\beta^0 &= \frac{\sigma^2}{\epsilon}, \quad \alpha^0 = 0, \quad \psi^0 = 0.
\end{aligned} \tag{5}$$

Figure 2 shows that, after passage through a region with Stern-Gerlach deflecting quadrupoles, the beta functions in both planes expand through a drift space, as if from a point source, from a low beta waist to a downstream collimating lens that produces a more or less parallel beam.

Optical properties of the proposed beamline are shown in the following figures. To amplify the SG deflection the beam line needs to be as long as possible. This length is limited, however, by the requirement that the r.m.s. beam size remain small compared to the vacuum chamber radius. Since adiabatic dampling causes the beam emittances to shrink proportional to γ_e (for fixed q) this produces a $\gamma_e^{1/2}$ SG enhancement factor with increasing γ_e. This benefit "saturates" however, when the quadrupole strength required to produce the necessary focal length is no longer physically achievable. This limitation is expressed analytically below.

$N_c = 4$ cells for the FODO section are shown in Figure 2, but N_c could be increased with little effect on the matching. N_c is limited, however, by the fact that the same optics that magnifies the SG deflection also magnifies the sensitivity to transverse beam displacement injection error. (Operating on "integer resonance" only a finite number of cells can be tolerated.)

With the help of trim quadrupole at londitudinal coordinate $s = 0.4\,\mathrm{m}$, a solenoid at the beamline entry focuses the beam onto the SG section. Upstream optics is not shown. This optics amounts to injection into a periodic FODO lattice of arbitrary length. As shown in Figure 3, there are half quadrupoles at entrance and exit, but this is only for matching convenience in this note. To avoid the need for half quadrupoles, Injection into a section of eight full-length alternating-gradient could be matched satisfactorily. From the point of view of Stern-Gerlach deflection each quadrupole acts as a thin element described completely by an inverse focal length q, precisely defined only for a thin element. But, from the point of view of orbit dynamics each quadrupole in the FODO section is a thick elements, with the orbit satisfying different differential equations in focusing and defocusing elements. Even so, the inverse focal length q defined in Eq. (4) provides the q-value needed to calculate the SG deflection.

Within the ELEGANT program, the focal lengths of the individual quadrupoles in the FODO line tuned for π phase advance per half cell are

$$q = kl_Q = 68.1\,\mathrm{m}^{-1}. \tag{6}$$

The corresponding focal length is $f = 0.0147\,\mathrm{m}$, which is about 0.3 times the full cell length, as seems roughly appropriate. Substitution of this q value and $l_Q = 0.02\,\mathrm{m}$ into Eq. (4) and

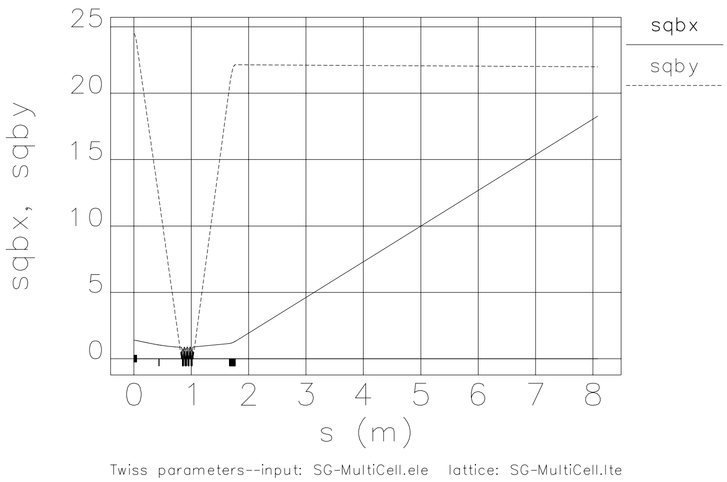

Figure 2: Beta functions for a full Stern-Gerlach detection beamline. SG displacements are measured at the end of the beam line. The length of the beamline is limited by the requirement that the rms beam size is conservatively smaller than the vacuum chamber radius. At low beam energy the drift length following the SG deflection may therefore need to be shorter than indicated. An SG-detecting BPM, located at variable distance $L_{\rm drift}$ beyond the collimating lens at $s = 1.72\,$m, is not shown. With the optics shown, $L_{\rm drift} \leq 6.5\,$m.

rearranging produces

$$l_Q \langle \partial B_x/\partial x \rangle = \frac{68.1\,\mathrm{m}^{-1}}{587\mathrm{T}^{-1}\mathrm{m}^{-1}}\,\gamma_e, \quad \text{or} \quad \left\langle \frac{\partial B_x}{\partial x} \right\rangle = \frac{68.1\,\mathrm{m}^{-1}}{0.02\,\mathrm{m} \times 587\mathrm{T}^{-1}\mathrm{m}^{-1}}\,\gamma_e = 5.80\,\mathrm{T/m}\,\gamma_e. \quad (7)$$

If the practical limit for $\partial B_x/\partial x$ is $500\,$T/m, then the apparatus being described could act as a Stern-Gerlach polarimeter up to $\gamma_e = 86$, or electron energy of $43\,$MeV.

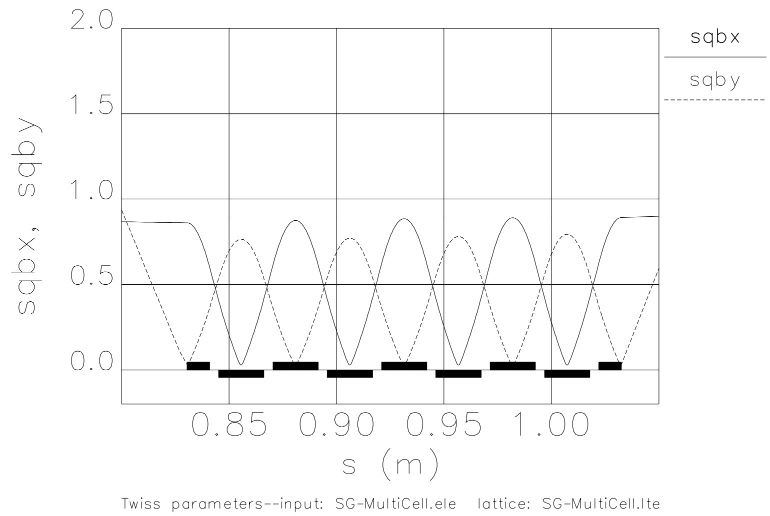

Figure 3: Optics in the periodic, SG deflection, multiple cell FODO lattice. The full quadrupole lengths are $l_Q = 2l = 0.02\,\mathrm{m}$ and the quad separation distances are $L = 0.005\,\mathrm{m}$. So the full cell length is $L_{\mathrm{cell}} = 0.05\,\mathrm{m}$.

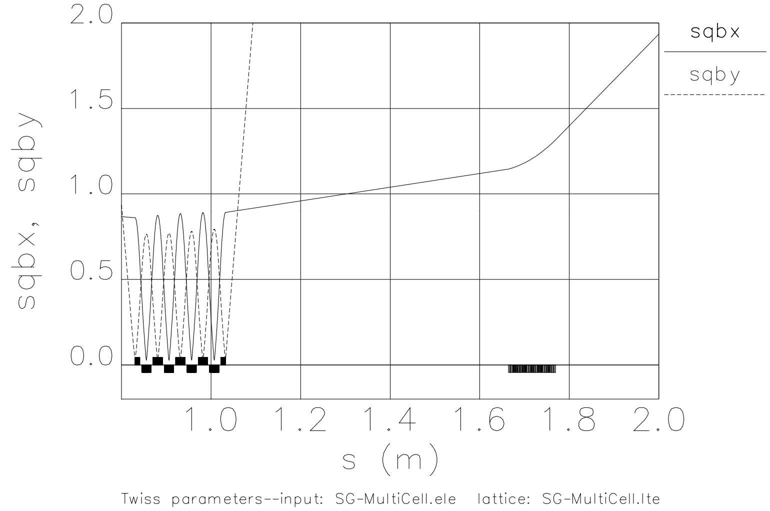

Figure 4: The quadrupole at $s = 1.72\,\mathrm{m}$ (at a distance $L_{\mathrm{coll.}} = 0.8\,\mathrm{m}$ from the center of the FODO lattice) is needed to restrict the growth of the defocussed transverse coordinate. But it also has the beneficial effect of magnifying the SG deflection. At low electron energy the beam emittance may limit the exit drift length to be shorter than shown to prevent beam loss before the beam passes through the BPM's.

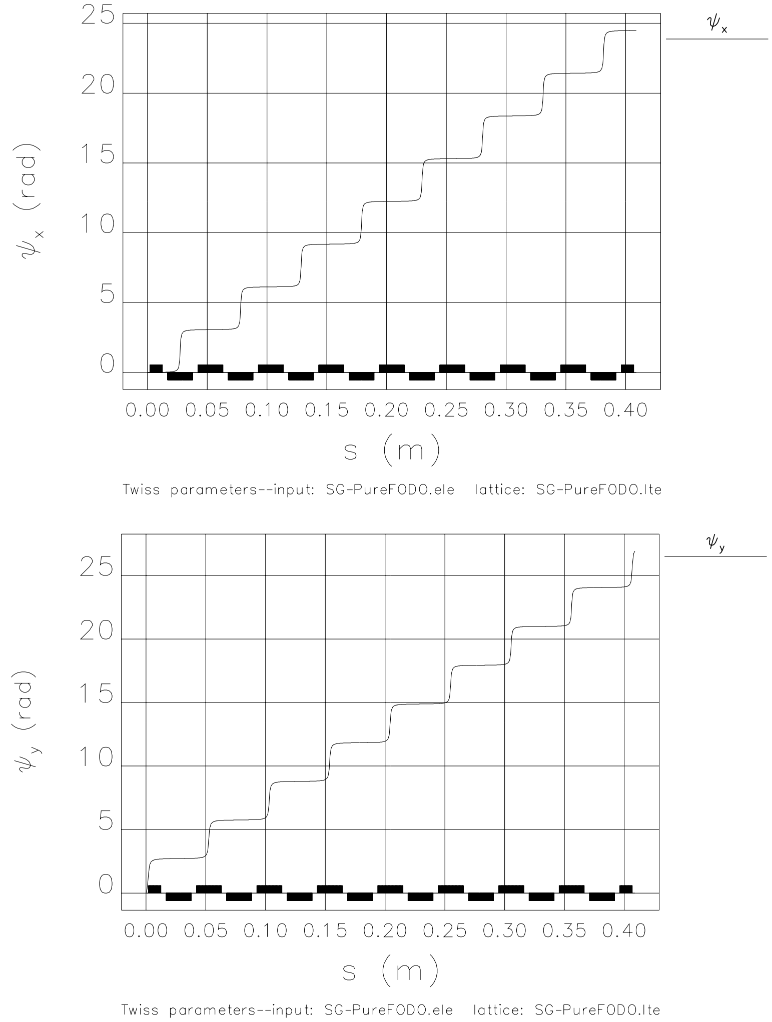

Figure 5: Phase advances ψ_x and ψ_y through a lattice with $N_c = 8$ cells. Since $25/8 = 3.125$, one sees that the phase advances per half cell are quite close to the value of 180 degrees, the maximum value that could be stable for arbitrarily large value of N_c. It is also the value for which all SG deflections superimpose constructively.

3 Analytic lattice model

The permanent magnet FODO lattice can be described analytically in closed form, following Steffen[4]. According to his Eqs. (1-15a) and (1-16a), the transfer matrices through uniform focusing and defocusing quadrupoles are given, respectively by

$$\begin{pmatrix} y \\ y' \end{pmatrix}_l = \begin{pmatrix} \sin\phi & \sin\phi/\sqrt{k} \\ -\sqrt{k}\sin\phi & \cos\phi \end{pmatrix}_0 \begin{pmatrix} y \\ y' \end{pmatrix}_0 = \mathbf{M}^+ \begin{pmatrix} y \\ y' \end{pmatrix}_0$$
$$\begin{pmatrix} y \\ y' \end{pmatrix}_l = \begin{pmatrix} \cosh\phi & \sinh\phi/\sqrt{k} \\ \sqrt{k}\sinh\phi & \cosh\phi \end{pmatrix}_0 \begin{pmatrix} y \\ y' \end{pmatrix}_0 = \mathbf{M}^- \begin{pmatrix} y \\ y' \end{pmatrix}_0 \tag{8}$$

where

$$k = \frac{1}{pc/e}\frac{\partial cB_x}{\partial x}, \quad \text{and} \quad \phi = l\sqrt{k} > 0. \tag{9}$$

According to Steffen's Eq. (4-35), the transfer matrix through a half period is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{10}$$

where

$$\begin{aligned}
a &= \cos\phi\cosh\phi + \sin\phi\sinh\phi + \frac{L}{l}\phi\cos\phi\sinh\phi, \\
b &= \frac{l}{\phi}\Big(\cos\phi\sinh\phi + \sin\phi\cosh\phi + \frac{L}{l}\phi\cos\phi\cosh\phi\Big), \\
c &= \frac{\phi}{l}\Big(\cos\phi\sinh\phi - \sin\phi\cosh\phi - \frac{L}{l}\phi\sin\phi\sinh\phi\Big), \\
d &= \cos\phi\cosh\phi - \sin\phi\sinh\phi - \frac{L}{l}\phi\cos\phi\sinh\phi.
\end{aligned} \tag{11}$$

To produce constructive interference at every quad location the phase advance per half cell has to be π. Our parameter values, to produce 180 degree phase advance per half cell, are $l = 0.01$, $L = 0.005$, and $\phi = 3.141592654$. This produces $a = -29.73267065$, $b = -0.09472054548$, $c = -3628.143470$, $d = -11.59195327$. Other parameter values are $\tanh(\pi) = 0.9962720762$, determinant=1.0000002, $a/d = 2.564940520$. Other than the phase advance, which has been imposed "by hand", using ELEGANT, it is not clear at this time that these values agree with the preceeding curves, all of which were obtained by empirical fitting with ELEGANT.

4 Stern-Gerlach displacement

The magnetic field and Lorentz force vectors, in an "erect" quadrupole are indicated by solid arrows in Figure 6. The corresponding Stern-Gerlach force vectors, for beam polarized along the y-axis, are indicated by hollow arrows. The SG deflection in this case is horizontal, as the calculation on the right shows. The other calculation shows that horizontal polarization causes vertical SG deflection. In general, the SG deflection is at right angles to the beam

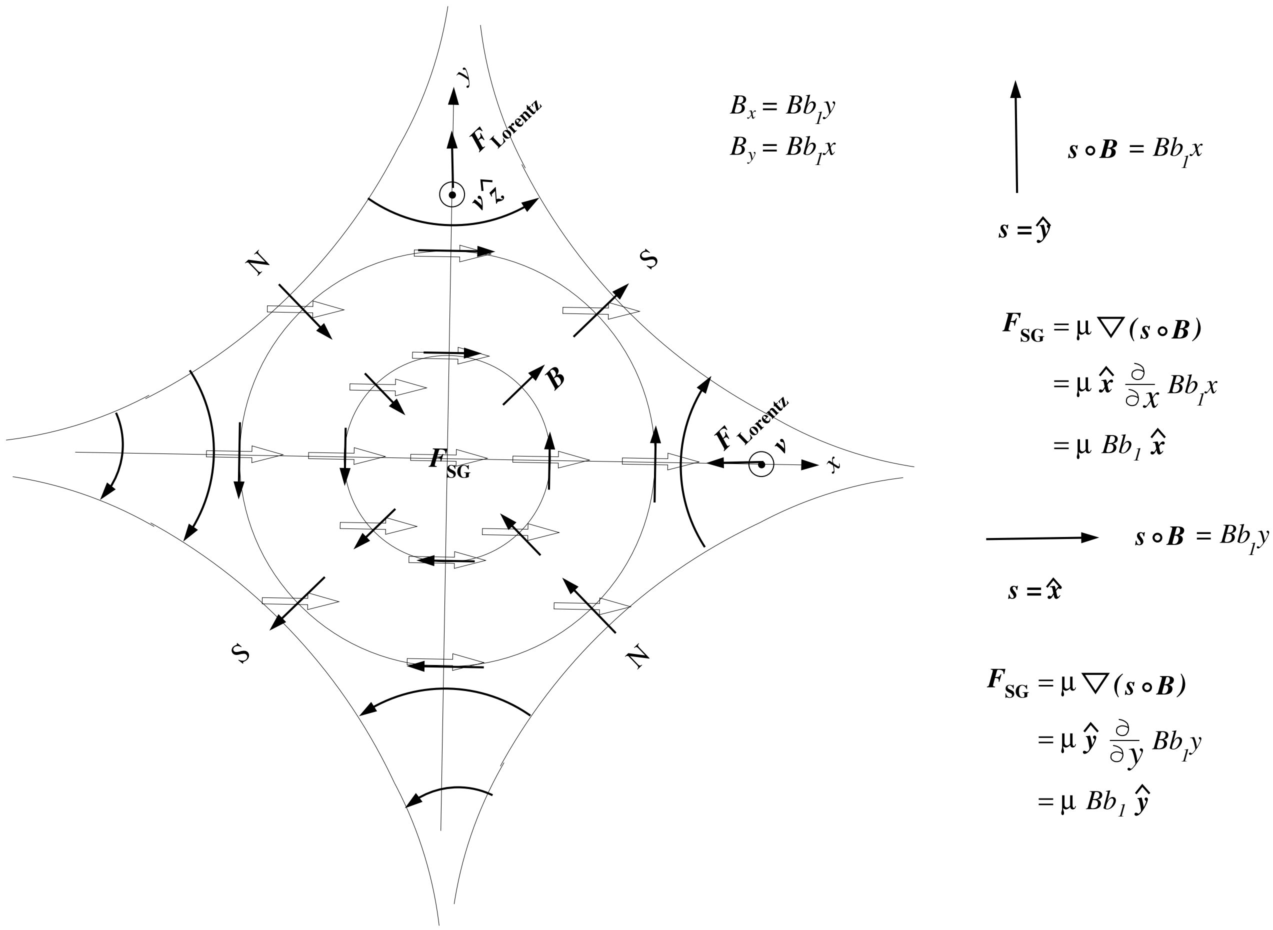

Figure 6: Magnetic field and Lorentz force vectors in an "erect" quadrupole are indicated by solid arrows. The corresponding Stern-Gerlach force vectors, for beam polarized along the y-axis, are indicated by hollow arrows.

polarization, irrespective of the quadrupole orientation. The ratio of Stern-Gerlach force to electromagnetic force is determined by a ratio of coupling constants:

$$\frac{\mu_B/c}{e} = 1.930796 \times 10^{-13}\,\mathrm{m}, \tag{12}$$

where, except for anomalous magnetic moment and sign, Bohr magneton μ_B is the electron magnetic moment. The Stern-Gerlach deflection in a quadrupole is strictly proportional to the inverse focal lengths of the quadrupole;

$$\Delta\theta_y^{SG} = \frac{\mu_x^*}{ec\beta} q_x, \quad \text{and} \quad \Delta\theta_y^{SG} = \frac{\mu_y^*}{ec\beta} q_y. \tag{13}$$

The magnetic moments μ_x^* and μ_y^* differ from the Bohr magnetron μ_B only by $\sin\theta$ and $\cos\theta$ factors respectively As already stated the electron velocity is being set exactly to c. The SG deflection at fixed magnet excitation is then proportional to $1/\gamma$. Yet, superficially, these formulas show no *explicit* dependence on γ. This is only because the angular deflections are expressed in terms of quadrupole inverse focal lengths. For a given quadrupole at fixed quadrupole excitation, the inverse focal length scales as $1/\gamma$. Expressing the SG deflection in terms of inverse focal lengths therefore has the effect of "hiding" the $1/\gamma$ Stern-Gerlach deflection dependence, which comes from the beam stiffness.

For a single quadrupole, the Stern-Gerlach-induced angular deflection is

$$\Delta\theta^{SG} = (1.93 \times 10^{-13}\,\mathrm{m})\, q. \tag{14}$$

The transverse displacement Δx_j at downstream location "j" caused by angular displacement $\Delta\theta_{x,i}$ at upstream location "i" is given by

$$\Delta_{x,j} = q_x\,(1.93 \times 10^{-13}\,\mathrm{m})\,\sqrt{\beta_{x,j}\beta_{x,i}}\,\sin(\psi_{x,j} - \psi_{x,i}). \tag{15}$$

where $\psi_{x,j} - \psi_{x,i}$ is the horizontal betatron phase advance from "i" to "j". Rather than using this formula to determine the transverse deflection, it is simpler to note that the deflected orbit itself has no knowledge of beta functions, and use linear transfer matrix evolution;

$$\begin{pmatrix} \Delta x_{SG} \\ \cdot \end{pmatrix} = \begin{pmatrix} 1 & L_{\mathrm{drift}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1.49 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{coll}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \Delta\theta_{SG} \end{pmatrix}, \tag{16}$$

where the lengths L_{drift} and L_{coll} have been defined in preceeding figure captions, and the collimating quadrupole strength is 1.49 /m. Completing the matrix multiplication yields

$$\Delta x_{SG} = (0.8 + 2.19 L_{\mathrm{drift}})\Delta\theta_{SG}. \tag{17}$$

Combining this equation with Eq. (6), and including a factor $2N_c = 8$, to account for constructive interference from the multiple quadrupoles, the horizontal SG displacement is given by

$$\Delta x_{\mathrm{SG}} = \pm 2N_c\,(1.93 \times 10^{-13}\,\mathrm{m})\ \times 68.1\,\mathrm{m}^{-1}(0.8\,\mathrm{m} + 2.19 L_{\mathrm{drift}}) = 1.59 \times 10^{-9}\,\mathrm{m}. \tag{18}$$

The $\pm$ factor has the effect of doubling the SG displacement to 3.2 nm, because the BPM, which is tuned to half the bunch passage frequency, responds constructively to the oppositely polarized A and B beam bunches. We have gone to a huge effort to produced a 3.2 nm betatron deflection but, as discussed in previous notes, this should be measureably large enough to make Stern-Gerlach transverse electron polarimetry practical.

The Stern-Gerlach energy dependence has been much discussed in the past. The importance of the transverse beam size has not previously, as far as I know, been properly appreciated in those discussions. Expressing the quadrupole strength as an inverse focal length, as we have done, has had the effect of making the SG deflection independent of γ—this is just because the magnetic field has scaled proportional to γ. Once this scaling becomes impossible, the SG signal might seem to fall proportional to γ.

Even this conclusion is arguable, however. Actually the scaling is a bit more complicated. Even with the magnetic field gradient limited, the SG quadrupole lengths can be increased to preserve the optics described in this note, though with a longer FODO section.

As mentioned earlier the allowable drift length L_{drift} has been taken to be 6.5 m. But this length is, itself, a function of γ_e since the beam emittances are proportional to $1/\gamma_e$. So the actual scaling with energy is such that the maximum achievable Stern-Gerlach deflection increases as $\sqrt{\gamma}$ until the gradient can no longer be increased, and falls as $1/\sqrt{\gamma}$ as the electron energy is increased from there.

There is also a vertical SG deflection Δy_{SG}, but this is being neglected because it is demagnified by the optics.

As far as the proof-of-principle test at CEBAF, the most convenient energy depends more on real estate considerations for availabe test areas than on the SG energy dependence. Discussions so far have assumed $\gamma_e = 2$, 500 KeV electron kinetic energy, but this is for reasons of economy and accessibility, not because the SG signal is strongest at low energy. From Eq. (7), for the geometric parameters assumed in this not, the magnetic field gradient for $\gamma_e = 2$ would be 12 T/m, far less than the maximum possible.

References

[1] R. Li and P. Musumeci, *Single-Shot MeV Transmission Electron Microscopy with Picosecond Temporal Resolution,* Physical Review Applied 2, 024003, 2014.

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[3] Private communication from P. Musumeci.

[4] K. Steffen, *High Energy Beam OPtics,* Interscience, 1966