Quantum mechanical derivation of radio-frequency-driven coherent beam oscillations in storage rings

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The study of the approach to the quantum ground state and the possibility to detect displacements of macroscopic bodies close to the quantum limit represent pressing challenges in modern physics. In the recent experiment of the JEDI Collaboration at the COSY storage ring, the coherent oscillations of a deuteron beam were detected with an amplitude of only one order of magnitude above the limit of the Heisenberg uncertainty principle of about 40 nm for the one-particle betatron motion. On the other hand, the much discussed search for the permanent electric dipole moment of the proton with an ultimate sensitivity of 10^{-29} e cm requires control of the position of the beam center of gravity with an accuracy of ≈5 pm. In this paper, we develop the full quantum mechanical treatment of the coherent beam oscillations with ultrasmall amplitudes. In agreement with the Ehrenfest theorem, we find a continuity of the description of the coherent betatron motion from the large classical amplitudes down to the deep quantum region below the one-particle Heisenberg limit. We argue that quantum mechanics does not preclude control of the beam center with subpicometer accuracy.

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I. INTRODUCTION

One of the grand challenges in particle physics is the search for physics beyond the Standard Model (BSM). At the forefront of the high precision frontier is the proposed search for the electric dipole moment (EDM) of protons stored in an all-electric frozen-spin storage rings with a sensitivity of $d_p \sim 10^{-29}$ e cm that is some 15 orders of magnitude smaller than the magnetic dipole moment of the proton [1–4]. The primary motivation is that the experimental observation of a permanent EDM of any subatomic particle implies the explicit violation of time reflection (T) and parity (P) symmetries, and therefore, according to the CPT theorem, also involves the violation of CP in the flavor-preserving channel. The presence of the latter could shed light on the mystery of the anomalously large baryon asymmetry in the Universe, which vastly exceeds the expectations within the standard models of particle physics and cosmology [5,6].

Proper control of systematic effects encountered in the search for such a minuscule EDM requires concurrent measurements of the spin rotations of beams propagating in opposite directions in an all-electric ring. To achieve such an ambitious goal, one of the crucial tasks is to control the difference of the vertical positions of centroids of two beams along the orbit in the machine with an accuracy of about 5 pm [2]. One may wonder whether such a demanding accuracy is not prohibited by the Heisenberg uncertainty principle. Moreover, this accuracy is also in the range of the vertical displacement of the beams due to Earth's gravitational force. It should be noted that up till now such tiny gravitational effects were never considered in the design and construction of accelerators, but gravity causes an observable background in the searches for the proton EDM [7–10]. This issue is common to all proposals for EDM storage rings [1–4].

Recently, the first direct measurement of the amplitude of coherent oscillations in the micrometer range of an intense beam of deuterons in a storage ring, excited by an intentionally mismatched radio frequency (rf) Wien filter (WF) [11] at the Cooler Synchrotron COSY of Forschungszentrum Jülich has been reported [12]. A crucial point in the interpretation of this experiment is that, on timescales of duration of individual beam fills, the beam can be treated as a rarefied gas with weak intrabeam scattering. Moreover, within the framework of classical mechanics, the WF-driven oscillations of beam

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particles do have identical amplitudes and phases, regardless of the amplitude and phase of their idle betatron motion in the confining potential in the ring orbit. For this reason, with reference to the superposition principle, a solution of the one-particle problem provides an adequate description of the coherent WF-driven oscillations of the centroid of the beam. The achieved accuracy is about a factor of 10 larger than the amplitude of the zero-point betatron oscillation of *single* particles. The latter is ≈ 41 nm [see [12], Eq. (9)]. This should be compared with the actual size of the beam which is in the order of millimeter [12]. As we shall show, in the quantum mechanical analysis the same universality of the rf-driven oscillations results from the independence of the quantum-mechanical expectation value of the position operator, perturbed by interaction with the WF potential, from the quantum state of the individual particle in the confining potential in the ring.

The approach to the quantum ground state and the possibility of detecting displacements of macroscopic bodies near the quantum limit are the subject of intense theoretical and experimental efforts [13–17]. A notable example is the detection of gravitational waves with interferometric detectors using kilogram-scale mirrors, where the experimentally observed strains correspond to *subattometer-scale* displacements [14,18]. The case of ultrasmall coherent oscillations of particle bunches of rarefied gas confined in the focusing fields of storage rings complements and differs from the above examples. The very possibility of detecting small amplitudes arises from the fact that the signal of *coherent* oscillations is generated by $\approx 10^9$ particles contained in the bunch.

The subject of the present paper is the transition from the description within classical mechanics suitable for μm amplitudes in the COSY experiment [12] to the deep quantum regime of picometer amplitudes in the proposed ultimate proton EDM experiment. We start by illustrating the origin of the picometer domain in storage ring EDM experiments using the example of the derivation of the vertical beam displacement due to Earth's gravity in terms of the vertical betatron frequency. The possibility of subquantum amplitudes common to all particles in a bunch regardless of their quantum state in the potential which confines particles in the orbit was far from being obvious from the outset.

Our main conclusion is that the functional form of the amplitude of the coherent beam oscillation does not change during the transition from the classical to the quantum regime. We consider this point to be nontrivial, since amplitudes far below the Heisenberg uncertainty limit could have received large quantum corrections. We argue that it is not the case. As for the proposed dedicated proton EDM storage rings, it implies that *coherent* beam oscillations with amplitudes much smaller than the Heisenberg uncertainty limit for *single* particles are under good theoretical control. On the experimental side, the challenge lies in the sensitivity of the beam position monitors (BPM). Technical developments to achieve this goal are still in the early stages, but encouraging results have been reported in [19]. We also address the effects of intrabeam scattering and interactions with the residual gas on the coherent excitations of the beam.

II. VERTICAL SPLITTING OF COUNTERPROPAGATING BEAMS DUE TO EARTH'S GRAVITY

As an introduction into the subject, we explain how picometer-scale beam displacements emerge in the EDM experiment in storage rings. The attraction of Earth on the beam particles can not be switched off and must be compensated for by the focusing electromagnetic fields: either radial magnetic fields in the case of focusing by magnetic quadrupoles [3,4,7] or vertical electric fields in the case of electric focusing [1,2,8,9]. In usual storage rings, the ring plane and therefore the velocities of the circulating particles are orthogonal to the acceleration of free fall at Earth's surface, $\boldsymbol{g}_\oplus$. Under this condition, the force of the gravitational attraction on every stored particle of energy E, velocity $\mathbf{v}$, and rest mass m is given by [7–10,20–23]

$$\boldsymbol{F}_\mathrm{g} = \left(1+\frac{\mathbf{v}^2}{c^2}\right)\frac{E}{c^2}\boldsymbol{g}_\oplus = \frac{2\gamma^2-1}{\gamma}m\boldsymbol{g}_\oplus, \tag{1}$$

where $\gamma = 1/\sqrt{1-\mathbf{v}^2/c^2}$ is the pertinent Lorentz factor. A derivation of Eq. (1), based on the weak limit of the Schwarzschild metric in *isotropic* coordinates, was explained by Okun in Ref. [22]; see also Ref. [23] and extended discussion in Ref. [9].

The spring constant $\langle k\rangle$ of a confining oscillator potential can be related to the angular velocity ω_y of the vertical betatron oscillation as follows:

$$\langle k\rangle \approx \gamma m\omega_y^2. \tag{2}$$

Then gravity causes a downward displacement of the beam, which is given by

$$\Delta y \approx \frac{(2\gamma^2-1)|\boldsymbol{g}_\oplus|}{\gamma^2\nu_y^2\omega_\mathrm{rev}^2}, \tag{3}$$

where $\nu_y = \omega_y/\omega_\mathrm{rev}$ is the vertical betatron tune and ω_rev is the orbital angular velocity of the beam.

Of particular interest is the all-electric frozen-spin ring for counterrotating beams of protons with kinetic energy $T_p = 233$ MeV. The lattice for such a ring with focusing by electric quadrupoles, proposed in [1,2], anticipates a circumference of 500 m with $\omega_\mathrm{rev} = 2.26\times10^6$ s^{-1} and vertical betatron tune $\nu_y = 0.45$. The coherent vertical displacement of both beams induced by Earth's gravity is therefore

$$\Delta y_{\rm E} \approx 13\ \text{pm}. \tag{4}$$

In this scenario, both counterrotating beams will have an identical gravitational displacement. As for the spin rotations of the protons, the impact of the gravity-compensating electric field focusing in the vertical direction would correspond to a rotation of their magnetic dipole moments in a radial magnetic field of opposite sign for clockwise and counterclockwise propagating beams, respectively. This would distinguish such a signal from a real EDM signal, which will be identical for both beams. However small the displacement may be, the false EDM effect due to the gravity-compensating electric field is even larger than the EDM effect expected for $d_p = 10^{-29}$ e cm [2,8,9].

A recent proposal of a hybrid electric ring with magnetic focusing [3,4] assumes a circumference of 800 m with $\omega_{\rm rev} = 1.4\times 10^6\ \text{s}^{-1}$ and a vertical betatron tune $\nu_y = 2.3$, giving a coherent gravitational vertical displacement of

$$\Delta y_{\rm B} \approx 1.3\ \text{pm}, \tag{5}$$

and an average beam splitting of 2.6 pm. Note that the magnetic quadrupoles exert forces of opposite sign on counterrotating beams, resulting in vertically nonidentical orbits of the two beams.

III. CLASSICAL MECHANICS OF rf-DRIVEN BEAM OSCILLATIONS

The treatment of the classical limit provides the necessary background and elucidates the subsequent transition to the deep quantum regime. Here we follow the discussion of Ref. [12] and extend it to include the effects of intrabeam scattering and interaction with the residual gas. In the experiment described in Ref. [12], the beam was stroboscopically excited with a mismatched rf Wien filter once per turn. In this way, a vertical Lorentz force,

$$F_y(n) = F_y \cos(n\,\omega_{\rm WF} T), \tag{6}$$

is exerted on the stored particle, where n is the number of turns, $\omega_{\rm WF}$ denotes the angular phase velocity of the rf in the Wien filter, and $T = 2\pi/\omega_{\rm rev}$ is the beam revolution period. With vanishing Lorentz force, each individual particle of the beam performs idle vertical (and horizontal) betatron oscillations

$$y_{\rm idle}(t) = y(0)\sqrt{\frac{\beta_y(t)}{\beta_y(0)}}\cos\left[\psi_y(t)\right], \tag{7}$$

where $\beta_y(t)$ is the vertical betatron amplitude function. The amplitudes and initial betatron phases of different particles are uncorrelated. The betatron phase advance $\psi_y(t)$ satisfies $\psi_y(t+T) - \psi_y(t) = \omega_y T = 2\pi\nu_y$.

According to the superposition principle of solutions of linear ordinary differential equations, the generic betatron motion at the Wien filter position will be the sum

$$y(t)|_{t=nT} = [y_{\rm idle}(t) + y_{\rm driv}(t)]_{t=nT} \tag{8}$$

of the idle oscillation $y_{\rm idle}(t)$ of Eq. (7), the homogeneous solution, and the betatron motion $y_{\rm driv}(n) \equiv y_{\rm driv}(t)|_{nT}$ stroboscopically driven at times $t = nT$, $n > 0$, by the Wien filter in the straight section of the ring—the inhomogeneous solution with the initial condition $y_{\rm driv}(0) = 0$. The change at turn n of the vertical velocity v_y of the stored particle accumulated during the time interval $\Delta t = \ell/v_z \ll T$ spent by the particle with longitudinal velocity v_z per turn inside the Wien filter (which is of length ℓ short compared with the circumference of the storage ring) is given by

$$\Delta v_y(n) = \frac{F_y(n)\Delta t}{\gamma m} = -\zeta\omega_y \cos(n\omega_{\rm WF} T). \tag{9}$$

Again γ and m are the Lorentz factor and the rest mass of the particle, respectively, while the abbreviation $\zeta \equiv \frac{-F_y\Delta t}{\gamma m\omega_y}$ has been introduced here for convenience. The change Δy of the vertical position y in the short Wien filter can be neglected.

According to Eq. (7), the stroboscopic signal of the betatron motion observed at any point in the ring follows the harmonic law as a function of nT with angular velocity ω_y, and we invoke the familiar description of the oscillatory motion in terms of the complex variable $z = y_{\rm driv} - iv_y/\omega_y$. The one-particle master equation for the buildup of rf-driven oscillations directly downstream of the Wien filter is

$$z(n) = z(n-1)\exp(i\omega_y T) - \frac{i}{\omega_y}\Delta v_y(n). \tag{10}$$

Note that the contribution of the external force in this equation does not depend on the idle betatron motion of the particle. Subject to the initial condition $z(0) = 0$, Eq. (10) has the generic solution (valid for integer $n > 0$)

$$z(n) = -\frac{i}{\omega_y}\exp(i\omega_y nT)\sum_{k=1}^{n}\Delta v_y(k)\exp(-i\omega_y kT), \tag{11}$$

which yields for the stroboscopic force of Eq. (9)

$$z(n) = \frac{i\zeta}{2}\left[\frac{\exp(in\omega_y T) - \exp(in\,\omega_{\rm WF} T)}{\exp[i(\omega_y - \omega_{\rm WF})T] - 1} + \{\omega_{\rm WF} \to -\omega_{\rm WF}\}\right]. \tag{12}$$

This solution shows the standard phenomenon that every periodic external force also excites idle (homogeneous) oscillations $\propto \exp(in\omega_y T)$. A similar analytic result holds also for generic ac dipole-driven betatron oscillations, applied in a completely different context of machine diagnostics, described in Ref. [24] (see also references therein).

The amplitude of the rf Wien filter Fourier component of the beam oscillation, $y_{\mathrm{WF}}(n) = -\xi_y \cos(n\omega_{\mathrm{WF}}T)$, is given by

$$\xi_y = \frac{\zeta}{2} \frac{\sin(2\pi\nu_y)}{\cos(2\pi\nu_{\mathrm{WF}}) - \cos(2\pi\nu_y)}, \tag{13}$$

where the WF tune is $\nu_{\mathrm{WF}} = \omega_{\mathrm{WF}}/\omega_{\mathrm{rev}}$. Note the resonance at $\nu_{\mathrm{WF}} = \nu_y$.[1] The amplitude in Eq. (13) is independent of the amplitude and phase of the betatron idle motion of individual particles (7) and is shared by all particles in the bunch. It can be filtered out and measured by Fourier analysis of the BPM response with, e.g., lock-in amplifiers.

The Heisenberg uncertainty limit Q for the betatron oscillation amplitude ξ_y is obtained equating the betatron oscillation energy $\frac{1}{2}m\gamma Q^2\omega_y^2$ to the zero-point oscillator energy $\frac{1}{2}\hbar\omega_y$:

$$Q^2 = \frac{\hbar}{m\gamma\omega_y}. \tag{14}$$

Under the conditions of the experiment, this gives

$$Q = \frac{82}{\sqrt{\gamma\nu_y}} \mathrm{nm} = 41 \text{ nm}, \tag{15}$$

while the smallest value of the *measured* oscillation amplitude at BPM17 of COSY was $\xi_y^{\min}|_{\mathrm{BPM}} = (1.08 \pm 0.52)$ μm, so that at the location of the Wien filter $\xi_y^{\min}|_{\mathrm{WF}} = (0.45 \pm 0.22)$ μm was deduced, as described in Ref. [12].

One can extend the above considerations to include the impact of the interaction with the residual gas (RG). Let us assume that the scattering off the residual gas occurs during the turn n_{RG}, followed by an instantaneous vertical velocity kick v_y^{RG}, where the particle still remains within the ring acceptance angle, θ_{acc}. It is then straightforward to add the contribution of such a collision-driven kick to the generic expansion (11) by encoding it as a further $-i\Delta v_y/\omega_y$ contribution. The result is given by

$$z^{\mathrm{RG}}(t)|_{t=nT} = -\frac{i}{\omega_y} v_y^{\mathrm{RG}} \exp\left[i\{\omega_y(n - n_{\mathrm{RG}})T + \psi_y^{\mathrm{RG}}\}\right] \times \Theta[n - (n_{\mathrm{RG}} - \epsilon)], \tag{16}$$

where ψ_y^{RG} is the betatron phase advance from the scattering point to the WF, while Θ is the Heaviside step function with $\epsilon > 0$ and infinitesimal. According to the superposition principle, this contribution has to be added linearly to the betatron oscillation (8), $y(t)|_{t=nT} \to [y(t) + y^{\mathrm{RG}}(t)]_{t=nT}$ where $y^{\mathrm{RG}}(t)|_{t=nT}$ is the real part of (16). Indeed, the oscillation pattern shows that scattering off the residual gas does not contribute at all to the rf-driven oscillations with angular phase velocity ω_{WF}. Such a kick only changes the amplitude and phase of the idle betatron motion of a particle, thus contributing to the emittance growth, but leaves the evolution of the rf-driven oscillations unimpeded. The same is true, of course, for the intrabeam Coulomb scattering.

However, weak though they are, losses due to intrabeam scattering and elastic scattering off the residual gas beyond the ring acceptance angle, and the absorption due to inelastic electromagnetic and hadronic interactions with the residual gas must be taken into account. To this end, the EDM rings discussed in the literature [1–4] are expected to operate with revolution frequencies in the ballpark of 1 MHz, so that typically $T/\tau \sim 10^{-9}$ (given that $\tau \sim 10^3$ s, see, e.g., Fig. 14 in [12], and $\tau > 6000$ s in [25]). Consequently, the familiar quadratic attenuation correction to the betatron oscillation can in principle be neglected, as well as damping for any single turn—the damping becomes observable upon accumulation after a very large number of turns beyond $n \sim 10^6$.

The effects of beam damping can be addressed as follows. Note that the experimentally observable quantity at a given time t is the number of unabsorbed particles in the beam times the amplitude of the rf-driven oscillations, which is *universal* to *all* unabsorbed particles in a WF-passing beam bunch. Therefore, the rf-driven oscillations of a nonabsorbed particle cannot be affected at all by the attenuation at times prior to the observation time t. The corollary is that during the continuous operation of the Wien filter, regardless of the number of particles in the unabsorbed bunch, the observed *coherent* oscillation amplitude of the centroid of the bunch, $y_{\mathrm{WF}}(n) = -\xi_y \cos(n\omega_{\mathrm{WF}}T)$, is preserved and can still be extracted from Eq. (12).

IV. QUANTUM MECHANICS OF rf WIEN FILTER-DRIVEN OSCILLATIONS

The above treatment of the proximity to the quantum limit by classical mechanics can be justified *a posteriori*, should the observed amplitudes turn out to be much larger than the amplitude of the zero-point quantum oscillations, as was the case in Ref. [12] [see also Eq. (15)]. Such *a posteriori* comparison of two amplitudes could have been performed the other way around if the perturbation of the beam had been much smaller than the zero-point quantum amplitude. The challenge is to demonstrate that all particles

[1] In fact, the real part of $z(n)$, as given in Eq. (12), simply reads $y_{\mathrm{driv}}(n) = -\xi_y\{\cos(n\omega_{\mathrm{WF}}T) - \cos(n\omega_y T)\} + \frac{\zeta}{2}\sin(n\omega_y T)$. Thus on resonance, we obtain $y_{\mathrm{driv}}^{\mathrm{res}}(n) = -\frac{\zeta}{2}(n-1)\sin(n\omega_{\mathrm{WF}}T)$.

in a rarefied-gas beam do acquire identical rf-driven oscillation amplitudes regardless of their quantum state. In order to treat this new regime, one has to resort to the time-dependent Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = \{\hat{H}_0 + \hat{V}(t)\}\Psi(t), \tag{17}$$

where $\hat{H}_0$ is the time-independent Hamilton operator of the harmonic oscillator (HO), while the perturbative potential reads

$$\begin{aligned}\hat{V}(t) &= -F_y \cdot \hat{y} \cdot \cos(\omega_{\mathrm{WF}} t) \\ &= -F_y \cdot \sqrt{\frac{\hbar}{2\gamma m \omega_y}}(a^\dagger + a) \cdot \cos(\omega_{\mathrm{WF}} t) \\ &= -\frac{1}{\sqrt{2}} F_y Q (a^\dagger + a)\cos(\omega_{\mathrm{WF}} t),\end{aligned} \tag{18}$$

where $a^\dagger$ and a are the harmonic oscillator creation and annihilation operators.[2] This perturbation stroboscopically acts for very short time intervals, $\Delta t \ll T$, once per turn at time $t_n = nT$.

We are interested in ultrasmall oscillations driven by a weak rf potential, and resort to perturbative theory. Let $\Psi(-;n)$ be the wave function before, and $\Psi(+;n)$ directly behind the WF. The impact of the WF potential leads to the following discontinuity of the wave function:

$$i\hbar\{\Psi(+;n) - \Psi(-;n)\} = \hat{V}(nT)\Delta t \Psi(-;n). \tag{19}$$

This is a difference equation that takes the role of the differential equation (17) at this time step. It gives

$$\begin{aligned}&\Psi(+;n) \\ &= \left\{1 + i\frac{F_y\sqrt{\hbar}\Delta t(1-\delta_{n,0})}{\hbar\sqrt{2\gamma m\omega_y}}\cos(n\omega_{\mathrm{WF}}T)(a^\dagger + a)\right\}\Psi(-;n),\end{aligned} \tag{20}$$

which is valid for any positive integer n including $n = 0$, where the initial condition $\Psi(+;0) = \Psi(-;0)$ applies. The rest of the turn proceeds in the time-independent harmonic potential. The creation and annihilation operators change the energy of the state by $\pm\hbar\omega_y$ and thus its phase by $e^{\mp i\omega_y T}$, respectively. Taking this into account, we obtain the master equation

$$\begin{aligned}\Psi(-;k+1) = &\left\{1 + i\frac{F_y \Delta t(1-\delta_{k,0})}{\sqrt{2\hbar\gamma m\omega_y}}\cos(k\omega_{\mathrm{WF}}T)\right. \\ &\left.\times (a^\dagger e^{-i\omega_y T} + a e^{i\omega_y T})\right\}\Psi(-;k)e^{-i\omega_{\mathrm{in}}T},\end{aligned} \tag{21}$$

valid for any integer $k \geq 0$. Here $\hbar\omega_{\mathrm{in}}$ is the energy of the initial wave function $\Psi(-;0)$.

Before proceeding to a solution of this equation, we recall that the annihilation, a, and creation, $a^\dagger$, operators are dimensionless. Next we observe that the dimensionless parameter on the right-hand side of the master equation can be cast in the form

$$\eta = \frac{F_y \Delta t}{\sqrt{\hbar\gamma m\omega_y}} = \frac{\zeta}{Q}, \tag{22}$$

where the numerator is a classical mechanics scale of the oscillation amplitude in the perturbation potential [see Eq. (13)], while the denominator is the Heisenberg quantum uncertainty limit of Eq. (14). A perturbative solution of the master equation in the deep quantum regime of $\eta \ll 1$, which is the subject of this paper, proceeds as follows. The beam passes the Wien filter at times $t_k = kT$ with the integer k labeling the number of passes (= turns), i.e., $k = 1, 2, \ldots, n$. A transition from the initial state $|\Psi(-;0)\rangle \equiv |\Psi(+;0)\rangle = |\mathrm{in}\rangle$ to the perturbation components $a^\dagger|\mathrm{in}\rangle$ and $a|\mathrm{in}\rangle$ can take place during any pass k. According to Eqs. (20) and (21) the transition amplitude belonging to the component $a^\dagger|\mathrm{in}\rangle$ then acquires $\cos(k\omega_{\mathrm{WF}}T)$ from the Wien filter potential and, as an effect of the increase of energy by the creation operator by $\hbar\omega_y$, an extra phase factor $\exp[-i(n-k)\omega_y T]$ from the subsequent evolution in the confining potential. Here $k = n$ directly refers to Eq. (20), while the other cases $0 \leq k \leq n-1$ follow from Eq. (21), where $\Psi(-;1) = \Psi(-;0)e^{-i\omega_{\mathrm{in}}T}$ holds specifically for $k = 0$. The transition amplitude belonging to the component $a|\mathrm{in}\rangle$ acquires for the pass k again the factor $\cos(k\omega_{\mathrm{WF}}T)$ from the Wien filter potential but now the complex conjugate phase factor $\exp[i(n-k)\omega_y T]$ from the subsequent evolution in the confining potential. Upon the summation of all transitions from $k = 0$ to $k = n$, we obtain to linear order in η

$$\begin{aligned}&|\Psi(+;n)\rangle \\ &= \left\{1 + i\frac{F_y \Delta t}{\sqrt{2\hbar\gamma m\omega_y}}(S(n)a^\dagger + S^*(n)a)\right\}e^{-in\omega_{\mathrm{in}}T}|\mathrm{in}\rangle,\end{aligned} \tag{23}$$

[2]Note that force derived from the potential in Eq. (18), $\hat{F}_y(t) = -\frac{\partial\hat{V}(t)}{\partial\hat{y}} = F_y\cos(\omega_{\mathrm{WF}}t)$, is the quantum operator analog of the classical expression Eq. (6). Furthermore, $\hat{p}_y = i\sqrt{\frac{\hbar\gamma m\omega_y}{2}}(a^\dagger - a)$ is the canonical momentum operator of the position operator $\hat{y}$, such that the canonical commutator relation $[a, a^\dagger] = 1$ implies $[\hat{y}, \hat{p}_y] = i\hbar$ and vice versa.

valid for $n \geq 0$, where the accumulated complex-valued weight factor is given by

$$S(n) = (1-\delta_{n,0}) \sum_{k=1}^{n} \cos(k\omega_{\mathrm{WF}}T) \exp\{-i(n-k)\omega_y T\}$$
$$= \frac{1-\delta_{n,0}}{2} \left[\frac{\exp(-in\omega_y T) - \exp(-in\omega_{\mathrm{WF}}T)}{\exp(-i(\omega_y - \omega_{\mathrm{WF}})T) - 1} + \{\omega_{\mathrm{WF}} \to -\omega_{\mathrm{WF}}\} \right]. \tag{24}$$

Now we are in the position to evaluate the driven oscillation amplitude. To the linear order in the perturbation parameter $\eta \ll 1$, we obtain the WF-forced expectation value of the quantum mechanical position operator

$$y_{\mathrm{QM}}(n) = \sqrt{\frac{\hbar}{2\gamma m\omega_y}} \langle \Psi^*(+;n)|(a^\dagger + a)|\Psi(+;n)\rangle$$
$$= i\frac{F_y \Delta t}{2\gamma m\omega_y} \langle \mathrm{in}|[(a^\dagger + a), (S(n)a^\dagger + S^*(n)a)]|\mathrm{in}\rangle$$
$$= -i\frac{F_y \Delta t}{2\gamma m\omega_y} [S^*(n) - S(n)] \langle \mathrm{in}|[a, a^\dagger]|\mathrm{in}\rangle$$
$$= -i\frac{F_y \Delta t}{2\gamma m\omega_y} [S^*(n) - S(n)]. \tag{25}$$

We observe that the scale for $y_{\mathrm{QM}}(n)$ is set by the dimensionless perturbation parameter η times the Heisenberg uncertainty limit Q. And indeed,

$$\frac{F_y \Delta t}{\gamma m\omega_y} = \eta Q \tag{26}$$

yields $y_{\mathrm{QM}}(n)$, which, in view of Eq. (22), contains no explicit dependence on $\hbar$.[3]

It implies that the driven coherent oscillation amplitude in the deep quantum limit is universal for all particles in the beam, regardless of their individual quantum excited initial state, be it a pure state or a mixed state. The crucial point behind this insight is that the creation, $a^\dagger$, and annihilation, a, operators enter Eq. (25) in terms of the canonical commutator $[a, a^\dagger] = 1$. This is tantamount to the independence of the classical driven oscillation amplitude from the amplitude and phase of the inherent betatron motion [12]. The amplitude of the rf Wien filter-driven oscillations of a beam bunch is the same for each individual particle in the bunch, which is equivalent to the definition of a coherent oscillation of the beam. The analytic form of the derived quantum amplitude $y_{\mathrm{QM}}(n)$ of Eq. (25) is exactly the same as for $y_{\mathrm{driv}}(n) = \mathrm{Re}\, z(n)$ from Eq. (12), with ζ replaced by the ratio $\frac{-F_y \Delta t}{\gamma m\omega_y}$, cf. Eq. (9).

A comment on the mixed initial state is in order. Without loss of generality, we can assume that the density matrix of such a mixed state reads at turn n

$$\hat{\rho}(n) = \sum_{i=1}^{N} w_i |\Psi(+;n)_i\rangle\langle\Psi^*(+;n)_i| \tag{27}$$

with N real-valued weights $w_i \geq 0$ of normalization $\sum_{i=1}^{N} w_i = 1$. Note that each $|\Psi(+;n)_i\rangle$, labeled by the index $i = 1, \ldots, N$, evolves from its initial state $|\mathrm{in}_i\rangle$ of initial energy $\hbar\omega_{\mathrm{in}_i}$ in complete analogy to Eq. (23), with $S(n)$ exactly as in Eq. (24). Inserting $|\Psi(+;n)_i\rangle$ into the right-hand side of Eq. (25) we get, because of $\langle \mathrm{in}_i|(a^\dagger + a)|\mathrm{in}_i\rangle = 0$ and $\langle \mathrm{in}_i|[a, a^\dagger]|\mathrm{in}_i\rangle = 1$, still the final relation of Eq. (25). Thus the expectation value of $\hat{y}$ is given by the following trace relation:

$$\mathrm{Tr}[\hat{y}\,\hat{\rho}(n)] = \sum_{i}^{N} w_i \langle \Psi^*(+;n)_i|\hat{y}|\Psi(+;n)_i\rangle$$
$$= \sum_{i=1}^{N} w_i y_{\mathrm{QM}}(n) = y_{\mathrm{QM}}(n). \tag{28}$$

A comment on the higher order quantum corrections to the above result is in order. The second term, proportional to η, in the expansion of Eq. (23) is the first order correction of the unperturbed (initial) state $|\Psi_0(+;n)\rangle = e^{-in\omega_{\mathrm{in}}T}|\mathrm{in}\rangle$,

$$|\Psi_1(+;n)\rangle = i\frac{\eta}{\sqrt{2}}(S(n)a^\dagger + S^*(n)a)|\mathrm{in}\rangle e^{-in\omega_{\mathrm{in}}T}. \tag{29}$$

One of the typical η^2 corrections to the above derived first order term $y_{\mathrm{QM}}(n)$ is given by

$$y_2(n) = \langle \Psi_1(+;n)|\hat{y}|\Psi_1(+;n)\rangle$$
$$= \frac{\eta^2 Q}{2\sqrt{2}} \langle \mathrm{in}|(S^*(n)a + S(n)a^\dagger)(a^\dagger + a) \times (S(n)a^\dagger + S^*(n)a)|\mathrm{in}\rangle$$
$$= 0. \tag{30}$$

The reason for the vanishing corrections is that here emerges an expectation value of an operator, which contains an odd power of the annihilation operators a, unpaired with the creation operators $a^\dagger$, and vice versa. This generic argument holds for other second-order corrections, which stem from the $\propto \eta^2$ corrections to the initial wave function. This observation is sufficient for the purposes of our considerations. We can argue that our very special observable, which is an evidently odd function of the force F_y, can

[3]Note that $\sqrt{\hbar}$ cancels when the state (23) is inserted in the first relation of Eq. (25), subject to the initial condition $y_{\mathrm{QM}}(0) = \langle \mathrm{in}|\hat{V}(0)|\mathrm{in}\rangle = \langle \mathrm{in}|\hat{y}|\mathrm{in}\rangle = \langle \mathrm{in}|(a^\dagger + a)|\mathrm{in}\rangle = 0$, consistent with $y_{\mathrm{driv}}(0) = 0$ for the classical WF-driven component in Eq. (8).

receive corrections, if any, only from the third and higher odd orders of the perturbation theory.

Beam losses, arising from scattering beyond the beam acceptance angle or from direct absorption due to interactions with the residual gas, can be accounted for by augmenting the Schrödinger equation (17) with still another interaction $\hat{V}_{\rm ex}(t)$ which couples bound states to loss states,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = \{\hat{H}_0 + \hat{V}(t) + \hat{V}_{\rm ex}(t)\}\Phi(t). \qquad (31)$$

Within the standard Weisskopf-Wigner-Lee-Oheme-Yang formalism [26–29], Eq. (31) for the so extended ensemble of states Φ can be truncated back to the seemingly non-Hermitian equation for the bound states,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left\{\hat{H}_0 + \hat{V}(t) - \frac{i\hbar}{2\tau}\right\}\varphi(t). \qquad (32)$$

If one takes the ansatz

$$\varphi(t) = \Psi(t)\exp\left(-\frac{t}{2\tau}\right), \qquad (33)$$

one obtains from Eq. (32) with the attenuation term exactly the Schrödinger equation (17) for $\Psi(t)$. Accordingly, the above ansatz for $\varphi(t)$ admits an obvious interpretation: the exponential attenuation factor in

$$|\varphi(t)|^2 = |\Psi(t)|^2\exp\left(-\frac{t}{\tau}\right) \qquad (34)$$

describes the survival probability of unabsorbed beam particles, while $\Psi(t)$ describes the internal evolution of the unabsorbed state. Thus the expectation value of the operator $\hat{y}$ is still given by Eq. (25), which exactly agrees with $y_{\rm driv}(n)$. And this implies that the rf-induced *coherent* oscillations at $\omega_{\rm WF}$ of the centroid of the unabsorbed beam are still given by $y_{\rm WF}(n) = -\xi_y\cos(n\omega_{\rm WF}T)$ with input from Eqs. (12) and (13).

Finally, it should be noted that the above equality of the results of quantum mechanics for ultrasmall amplitudes in the deep quantum regime and classical mechanics for large amplitudes of driven oscillations can be regarded as an exemplary case of Ehrenfest's theorem. We reiterate that this result was not obvious from the beginning, since quantum corrections may have gained the upper hand in the deep quantum region. We presented arguments why this is not the case.

V. SUMMARY AND CONCLUSIONS

We have presented a quantum mechanical description of the excitation of coherent betatron oscillations by radio-frequency electromagnetic fields. Remarkably, one and the same formula [Eq. (13)] covers the whole range of amplitudes from large classical ones to well below the one-particle quantum limit. Neither scattering from the residual gas nor intrabeam scattering contribute to these coherent betatron oscillations of the beam centroid.

As far as the prospects of searches for the EDMs of charged particles in storage ring experiments are concerned, we conclude that in principle the amplitude of coherent oscillations of the center of mass of a particle bunch in a storage ring can be measured with an accuracy of better than one picometer within the framework of the Heisenberg uncertainty principle. Our analysis may be applied to other problems involving pulsed excitation of quantum oscillators.

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