Alpha-alpha scattering using Adiabatic Projection Method

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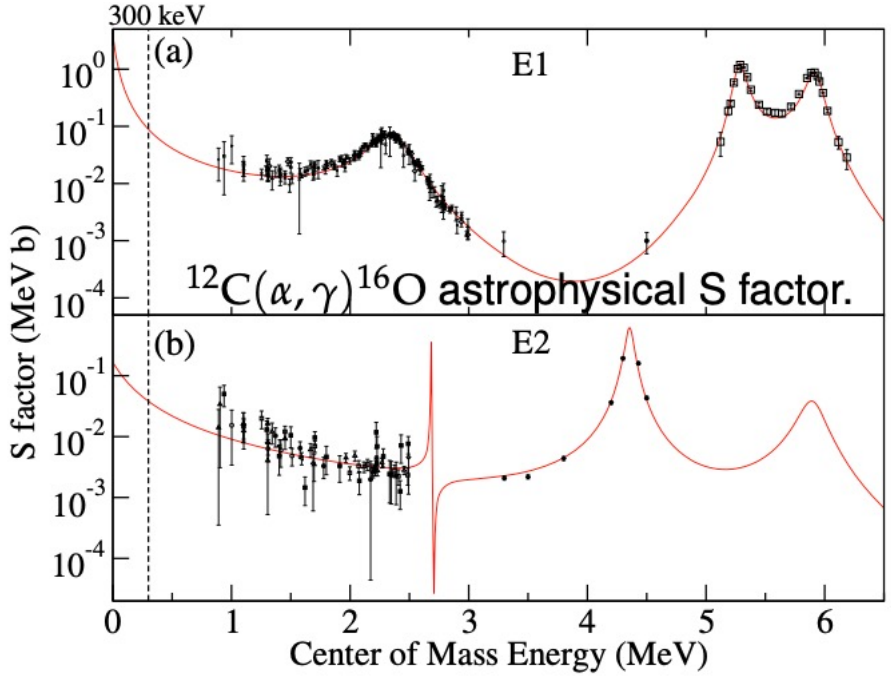
NLEFT Collaboration

Outline

- Introduction and Adiabatic Projection Method
- New calculation - Trimmed sampling
- Results
- Summary

Alpha-alpha scattering

- One of the most fundamental reactions in nuclear astrophysics.
- Alpha scattering, triple-alpha reaction, alpha capture play major role in stellar nucleosynthesis.
- Direct experimental results at 300 keV is impossible due to Coulomb barrier.
- For accurate reaction rate calculation, we need cross section within energy range 0.15 – 3.4 MeV.
- Study scattering by adiabatic projection method



Adiabatic Projection Method

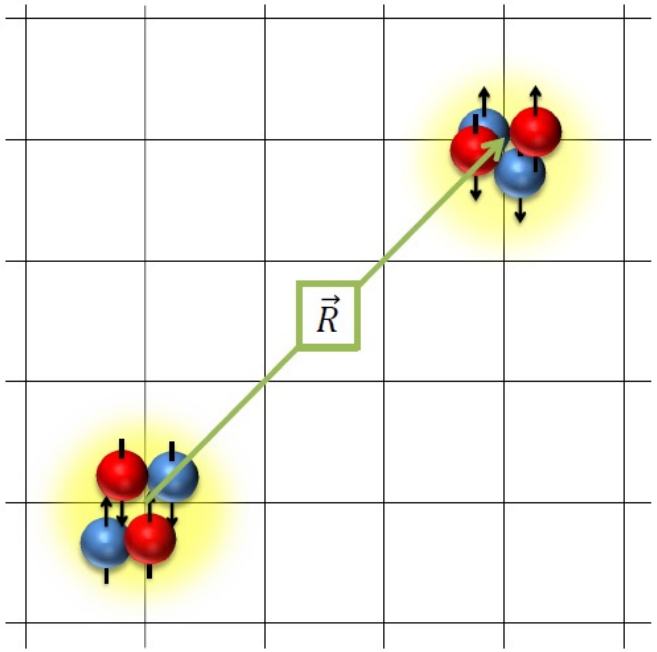
- Initial cluster states with relative separation $\vec{R}$

$$|\vec{R}\rangle = \sum_{\vec{r}} |\vec{r} + \vec{R}\rangle \otimes |\vec{r}\rangle$$

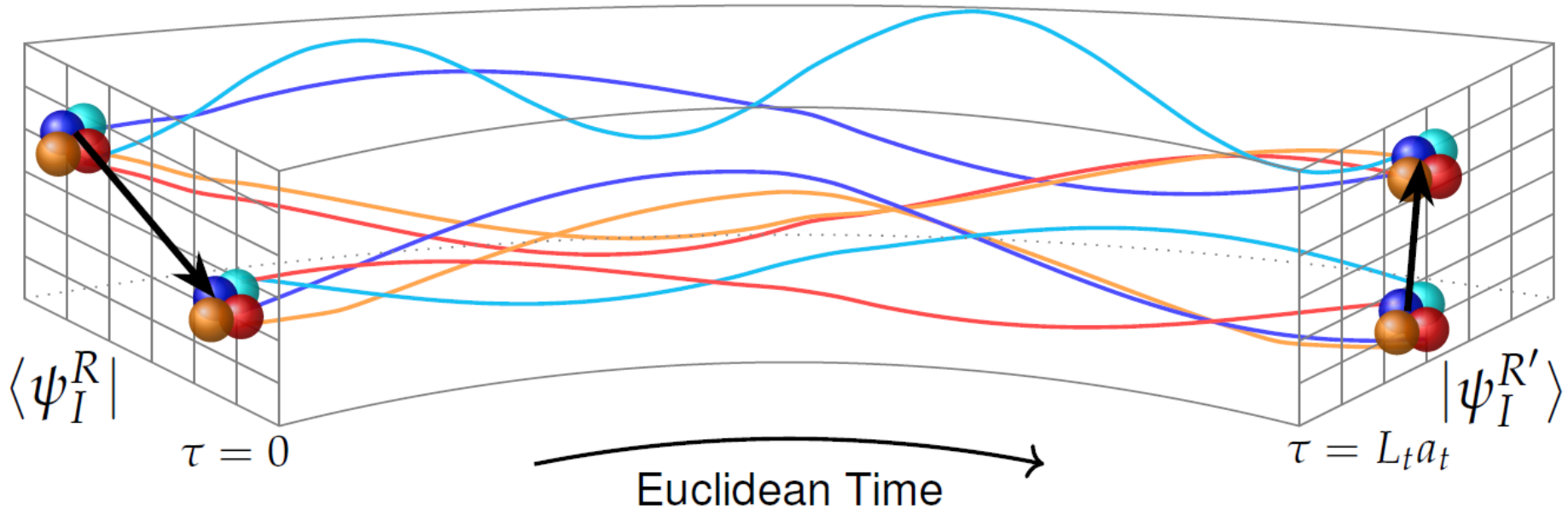
- Time evolved 'dressed cluster states'

$$|\vec{R}\rangle_\tau = \exp(-H\tau)|\vec{R}\rangle$$

  - H is full A-body interacting microscopic Hamiltonian
  - $\tau = L_t a_t$
  - $\lim_{\tau\to\infty} \left\{|\vec{R}\rangle_\tau\right\}$ spans low-energy two-cluster subspace



[1] M. Pine, D. Lee, G. Rupak, *Eur. Phys. J. A* 49, 151 (2013)
[2] S. Elhatisari, D. Lee, Phys. Rev. C 90, 064001
[3] S. Elhatisari, D. Lee, G. Rupak, E. Epelbaum, H. Krebs, T. Lähde, T. Luu, & U-G. Meißner. Nature 528, 111-114 (2015).
[4] S. Elhatisari, D. Lee, U-G. Meißner, G. Rupak, Eur. Phys. J. A 52, 174 (2016)
[5] S. Elhatisari, T. Lähde, D. Lee, U-G. Meißner, T. Vonk, *J. High Energ. Phys.* 2022



- Projected Hamiltonian

$$[H_\tau]_{\vec{R},\vec{R}'} = {}_\tau\langle \vec{R}|H|\vec{R}'\rangle_\tau$$

Norm

$$[N_\tau]_{\vec{R},\vec{R}'} = {}_\tau\langle \vec{R}|\vec{R}'\rangle_\tau$$

- Adiabatic Hamiltonian

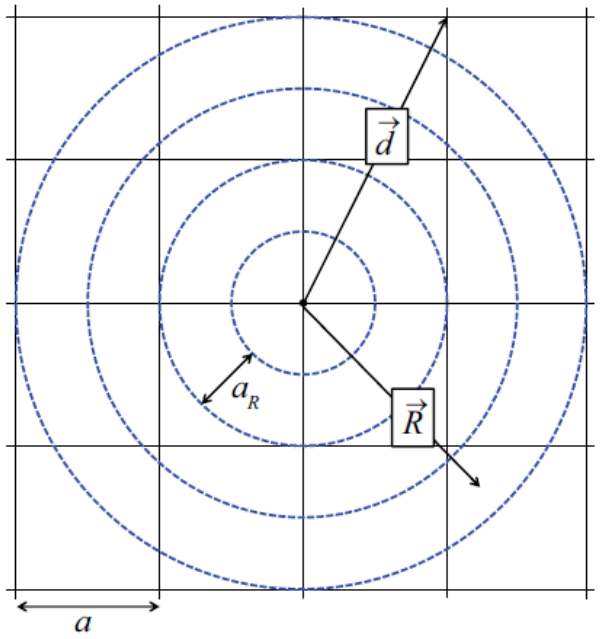
$$[H_\tau^a]_{\vec{R},\vec{R}'} = \left[N_\tau^{-1/2} H_\tau N_\tau^{-1/2}\right]_{\vec{R},\vec{R}'}$$

Adiabatic Hamiltonian

- 10x10 matrix → 100 elements to be computed → multi-channel calculations.
- Lowest eigenvalue → direct one-channel calculation (after infinite Euclidean time propagation).
- $E_{^8Be}(Lt = 20, N3LO) = -57.34 \pm 0.05$
  $E_{APM}(Lt = 20, N3LO) = -59.34 \pm 0.74$
- Information of the clusters at different distances → can extract phase shifts.
- Adiabatic projection method is much better than Lüscher's method [1]

[1] Lüscher, Comm. Math. Phys. 105 (1986) 153; NPB 354 (1991) 531

Radial binning



L	$[M^{\mathrm{a}}_{L_t}]_{\vec{d},\vec{d}'}$	$[M^{\mathrm{a}}_{L_t}]^{0,0}_{d,d'}$	$[M^{\mathrm{a}}_{L_t}]^{0,0}_{R,R'}$	
			$a_R = 0.125\,\mathrm{l.u.}$	$a_R = 0.250\,\mathrm{l.u.}$
10	$10^3 \times 10^3$	22×22	21×21	14×14
20	$20^3 \times 20^3$	85×85	58×58	34×34
30	$30^3 \times 30^3$	189×189	97×97	54×54
40	$40^3 \times 40^3$	335×335	137×137	74×74
50	$50^3 \times 50^3$	522×522	177×177	94×94
60	$60^3 \times 60^3$	752×752	217×217	114×114

- Project initial states on spherical harmonics

$$|\vec{d}\,\rangle = \sum_{\vec{n}} |\vec{n} + \vec{d}\,\rangle_1 \otimes |\vec{n}\,\rangle_2 \quad \Rightarrow \quad |d\rangle^{\ell,\ell_z} = \sum_{\vec{d}\,'} \mathrm{Y}_{\ell,\ell_z}(\hat{d}')\delta_{d,|\vec{d}\,'|}\,|\vec{d}\,'\rangle$$

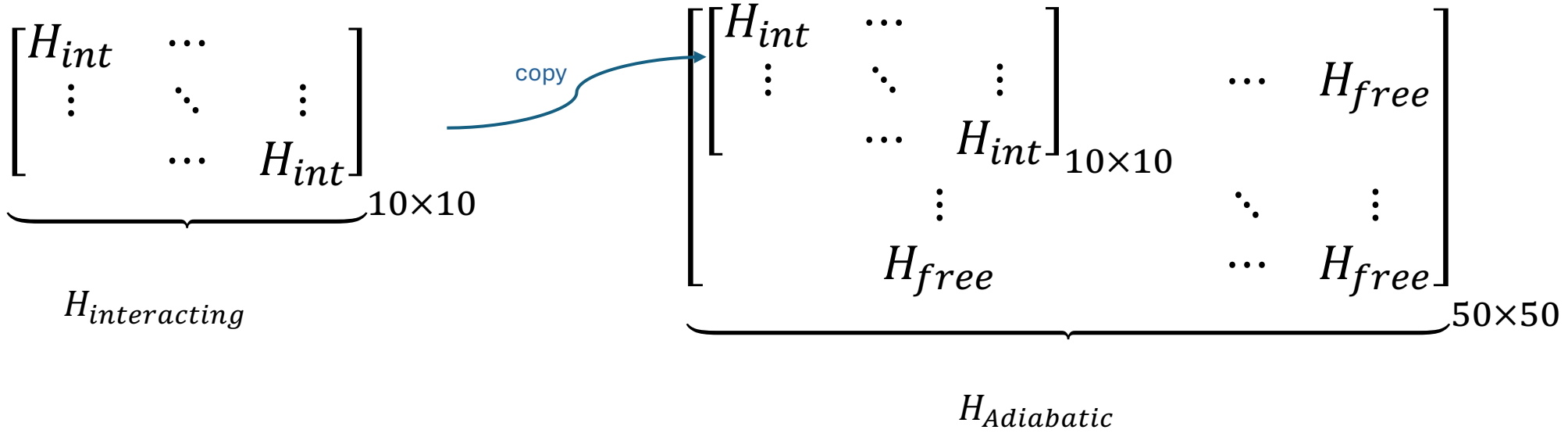
and bin the points together

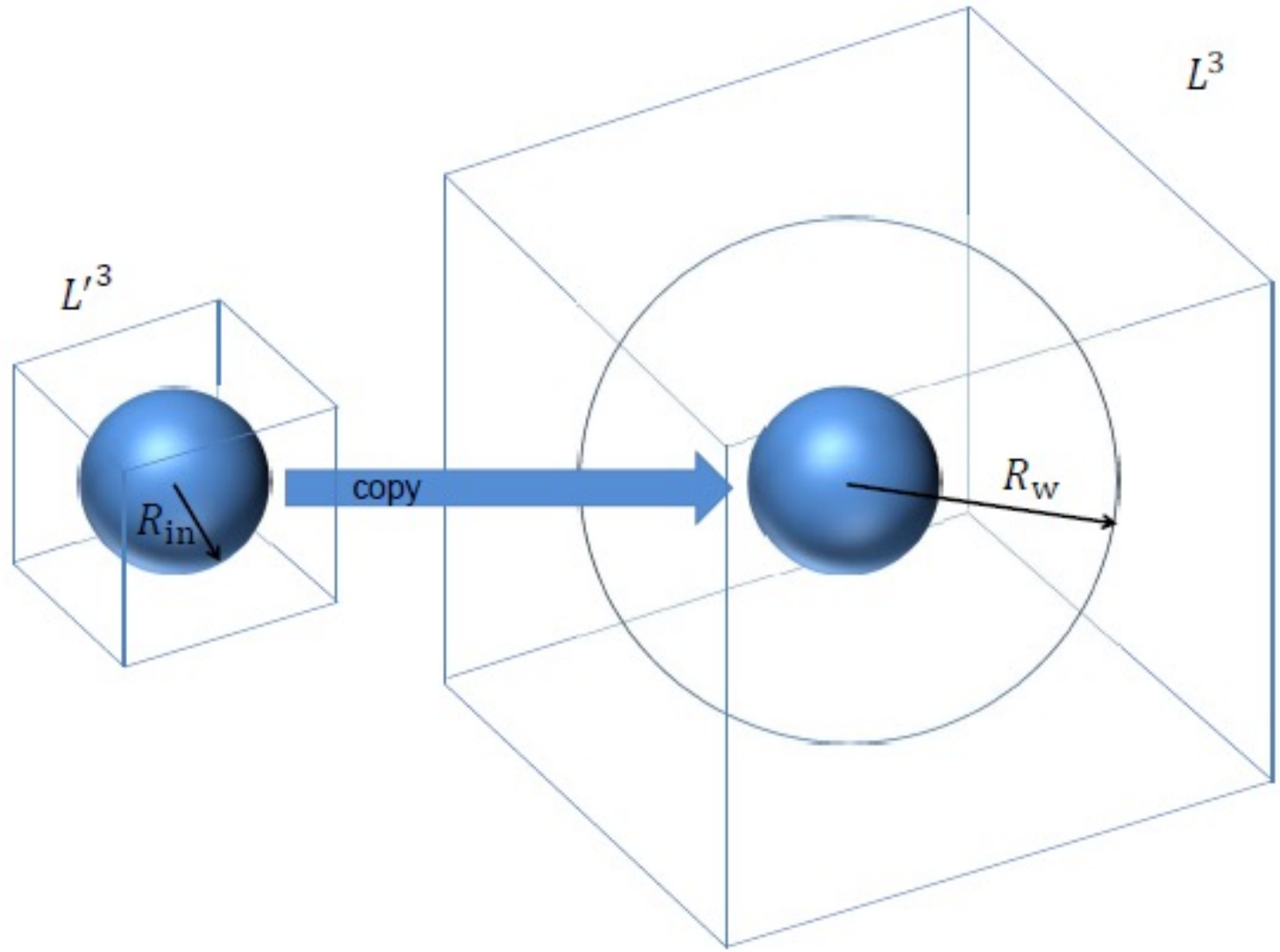
$$|R\rangle^{\ell,\ell_z} = \sum_{|d-R|<a_R/2} |d\rangle^{\ell,\ell_z}$$

[1] S. Elhatisari, D. Lee, U-G. Meißner, G. Rupak, Eur. Phys. J. A 52, 174 (2016)

Constructing large box Adiabatic Hamiltonian

- Low energy phase shifts → large lattice box size → full interacting calculation too expensive
- Two separate calculations – 1. Full interacting microscopic Hamiltonian in small box, 2. Non-interacting Hamiltonian in large box




L^3
L'^3
copy
R_{in}
R_{w}

Phase shift calculation

- Find eigenvector of Adiabatic Hamiltonian and look at asymptotic region

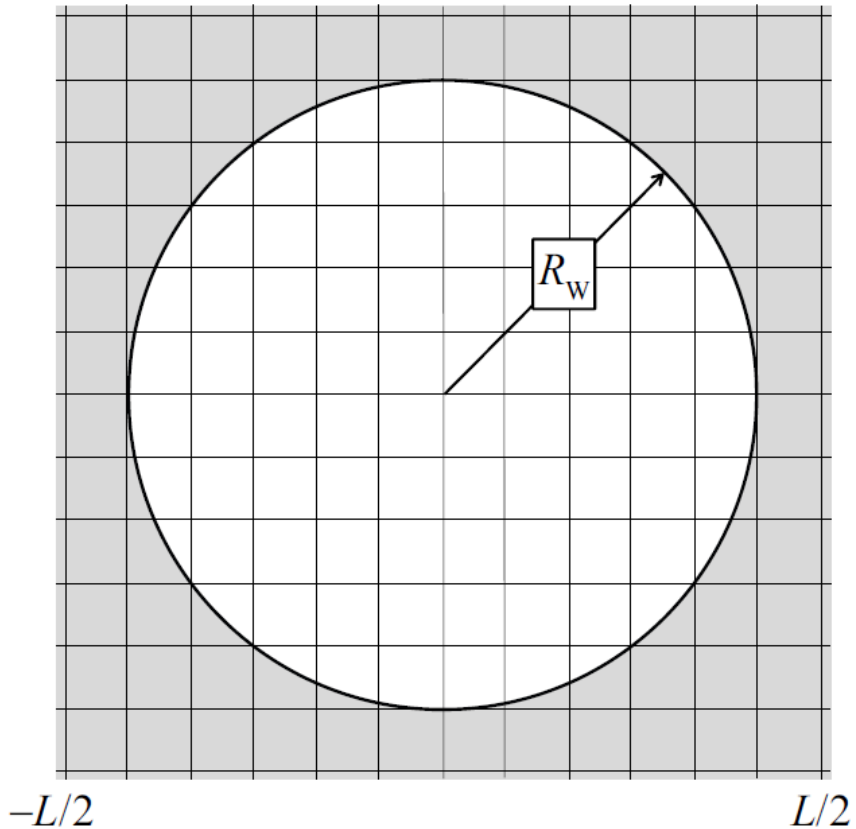
$$R_\ell^{(p)}(r) = N_\ell(p) \times \begin{cases} \cot\delta_\ell(p)\, j_\ell(p\,r) - n_\ell(p\,r), \\ \qquad \text{for any finite-range potential,} \\ \cot\delta_\ell(p)\, F_\ell(p\,r) + G_\ell(p\,r), \\ \qquad \text{for charged clusters,} \end{cases}$$

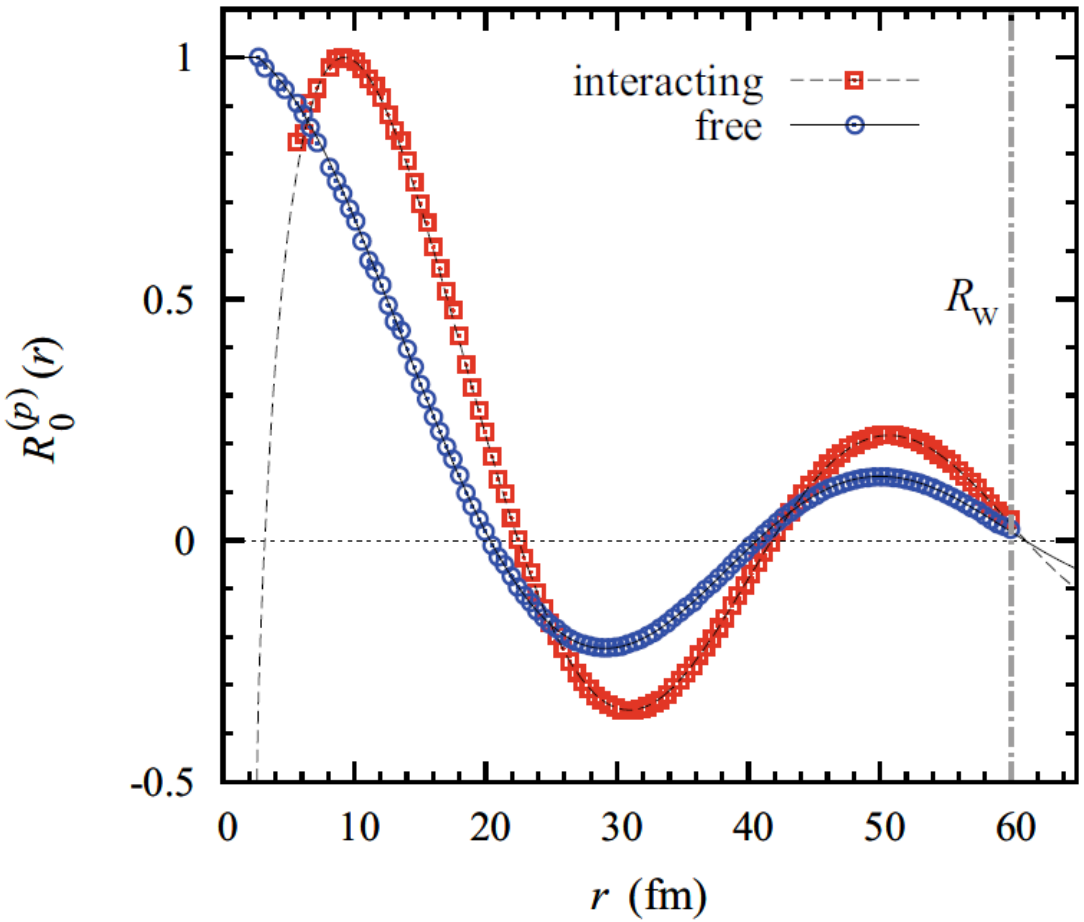
- Impose spherical hard wall and calculate phase shifts

$$\delta_\ell(p) = \begin{cases} \tan^{-1}\left[\dfrac{j_\ell(p\,R'_{\rm w})}{n_\ell(p\,R'_{\rm w})}\right], \\ \qquad \text{for any finite-range potential,} \\ -\tan^{-1}\left[\dfrac{F_\ell(p\,R'_{\rm w})}{G_\ell(p\,R'_{\rm w})}\right], \\ \qquad \text{for charged clusters.} \end{cases}$$

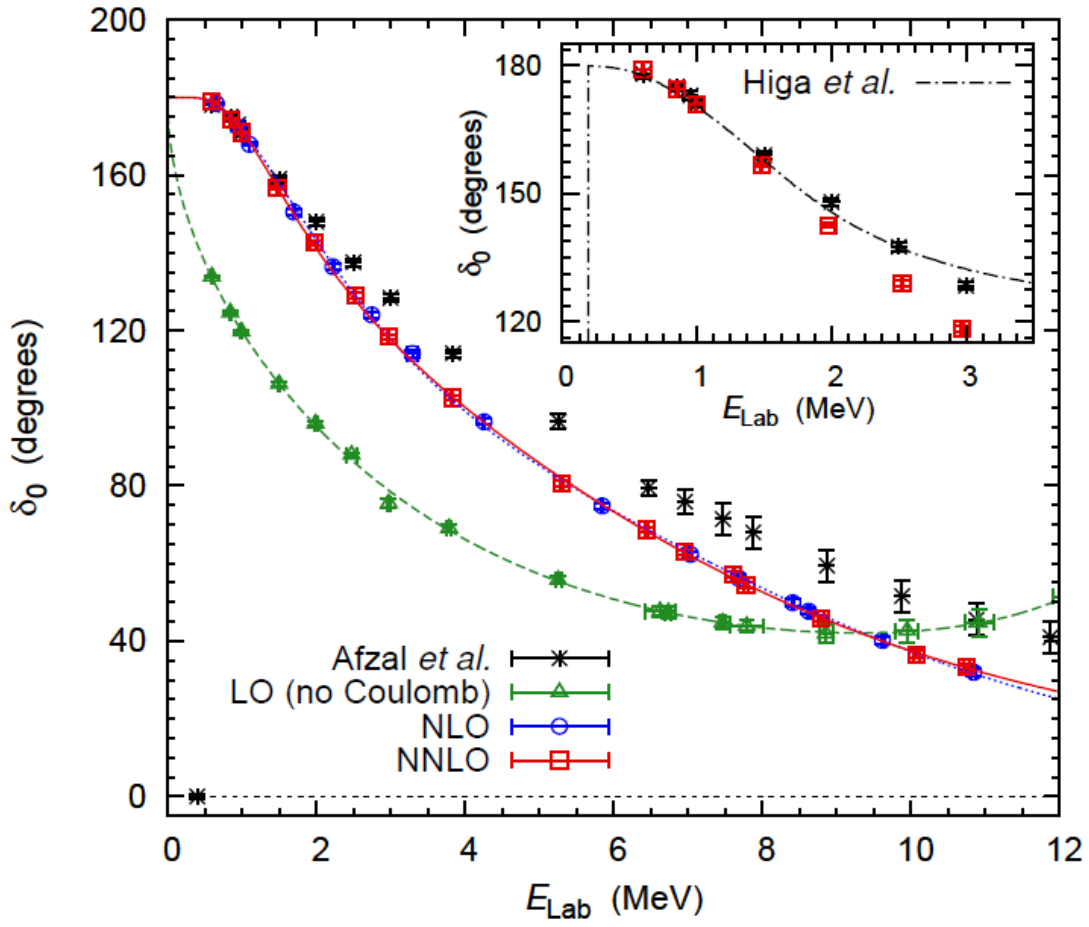
[1] J. Carlson, V.R. Pandharipande, R.B. Wiringa, Nucl. Phys. A 424, 47 (1984)
[2] B. Borasoy, E. Epelbaum, H. Krebs, D. Lee, U.-G. Meißner, Eur. Phys. J. A 34, 185 (2007)

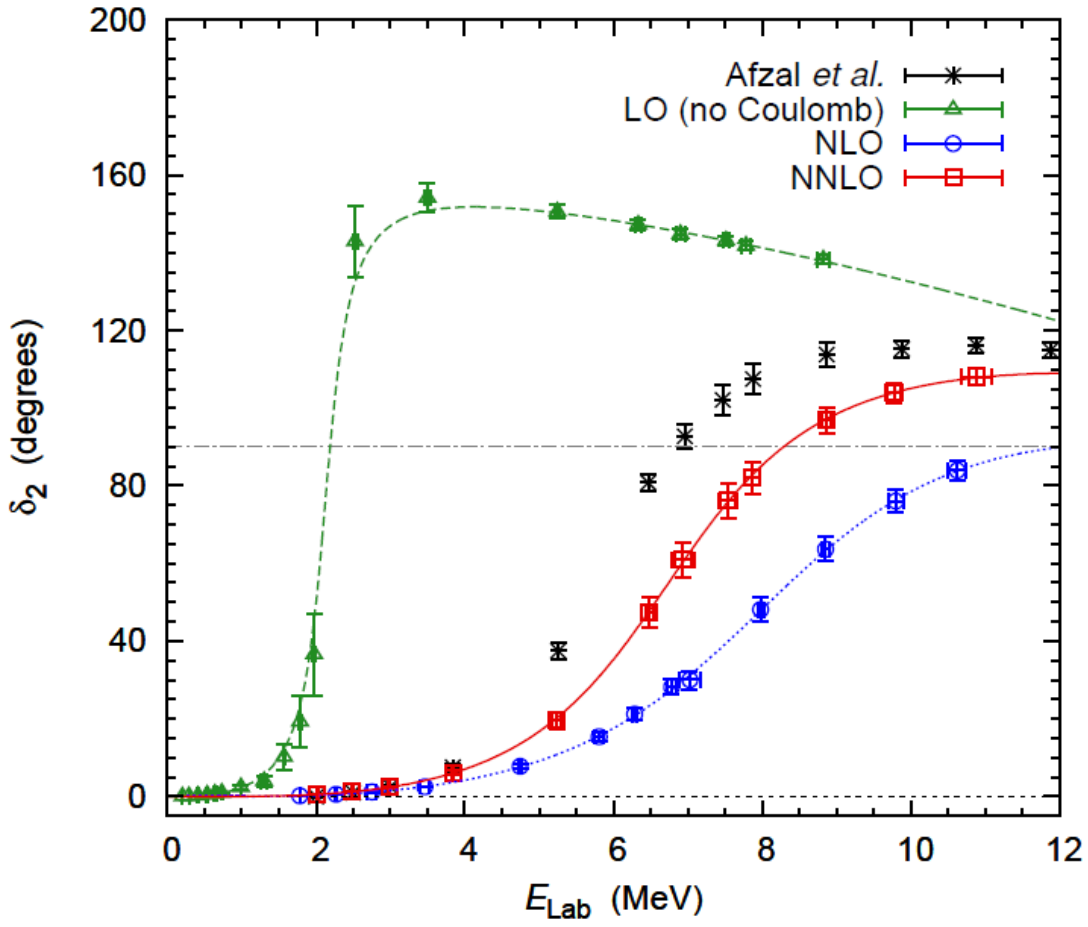
Spherical wall method





Previous alpha-alpha results



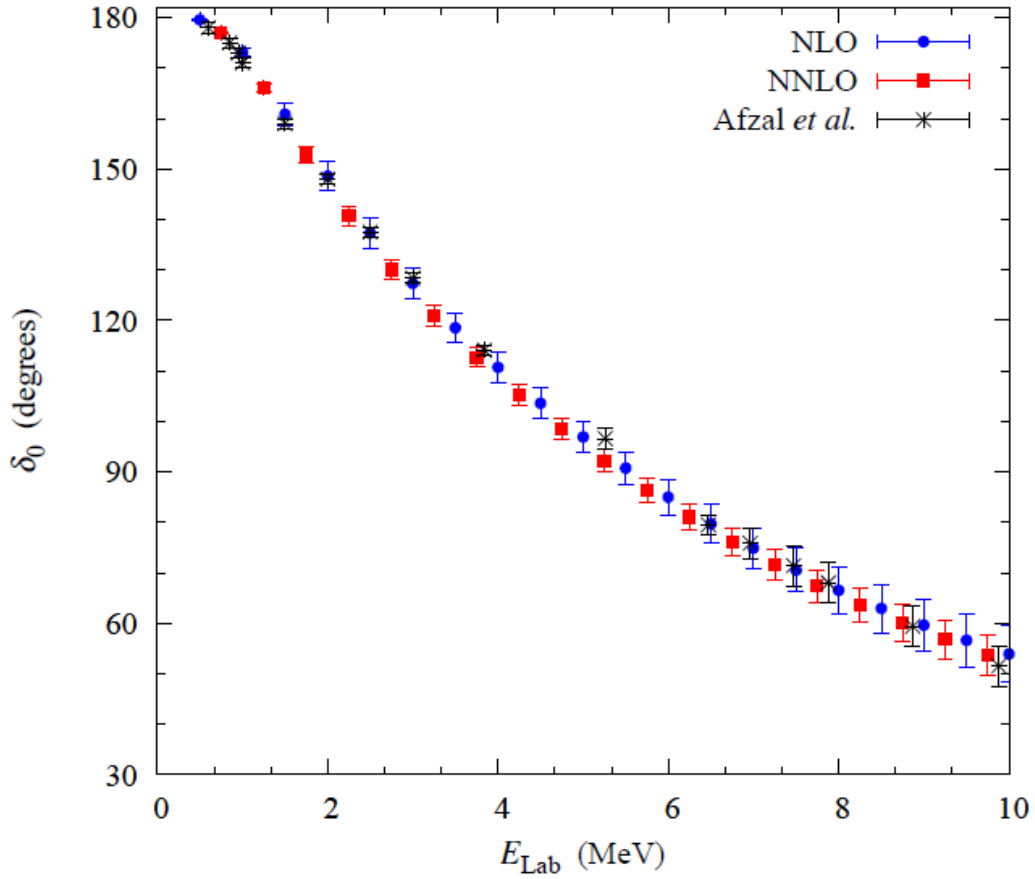


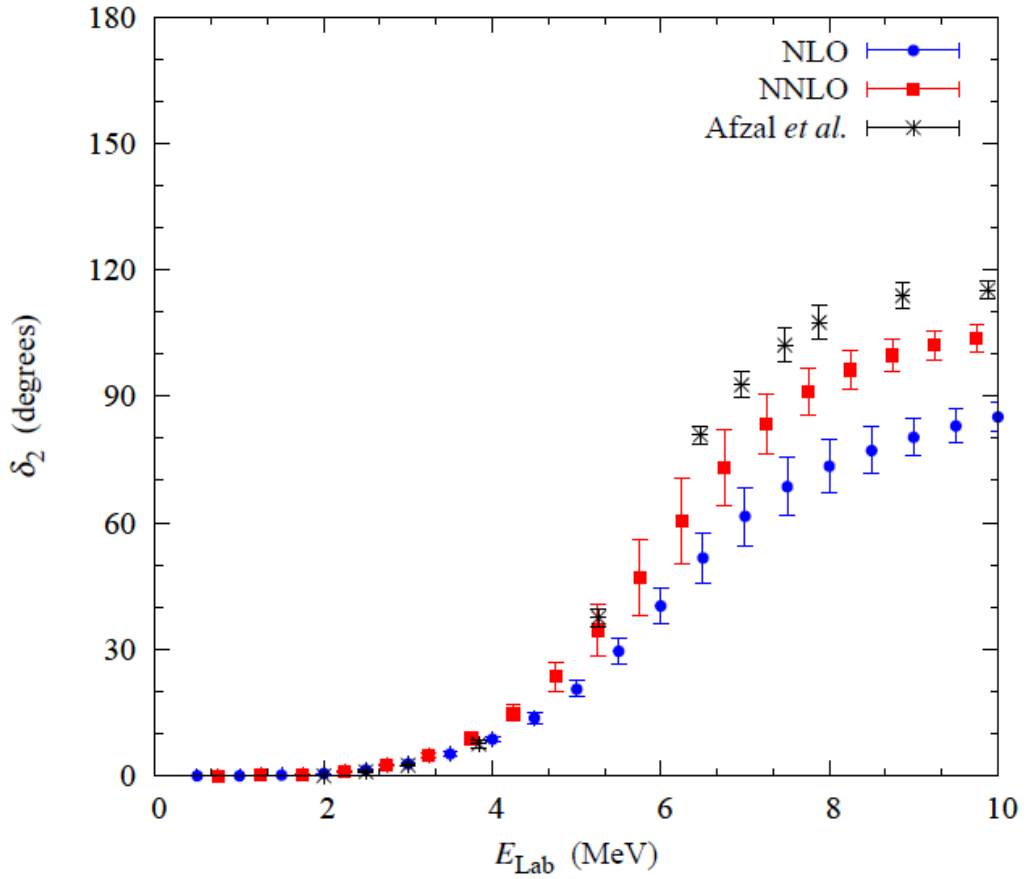
[1] Afzal, Ahmad, Ali, Rev. Mod. Phys. 41, 247, (1969)

[2] Higa, Hammer, van Kolck, Nucl.Phys. A809, 171 (2008), 0802.3426

[3] S. Elhatisari, D. Lee, G. Rupak, E. Epelbaum, H. Krebs, T. Lähde, T. Luu, & U-G. Meißner. Nature 528, 111-114 (2015).

More recent alpha-alpha results





[1] S. Elhatisari, T. Lähde, D. Lee, U-G. Meißner, T. Vonk, *J. High Energ. Phys.* 2022

This work vs previous work

- Smaller lattice spacing – 1.97 *fm* vs 1.32 *fm*
- Smaller temporal spacing – 1.32 *fm* vs 0.2 *fm*
- Improved interaction – NNLO vs N3LO
- Wavefunction matching

Challenges

- Expensive calculation
- Noise sensitive

Noise sensitive

$$E_{^8Be}(Lt = 20, N3LO) = -57.34 \pm 0.05$$

$$E_{APM}(Lt = 20, N3LO) = -69.21\ (+10.70\ -\ 33.42)$$

More statistics → $E_{APM}(Lt = 20, N3LO) = -59.34 \pm 0.74$

$$\begin{pmatrix} H_{11} & \cdots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \cdots & H_{nn} \end{pmatrix} \longrightarrow \text{Noisy matrices}$$

$H_{Adiabatic}$

Larger Lt requires even more statistics

Sampling

$$\begin{pmatrix} H_{11}+\delta H_{11} & \cdots & H_{1n}+\delta H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1}+\delta H_{n1} & \cdots & H_{nn}+\delta H_{nn} \end{pmatrix} \qquad \begin{pmatrix} N_{11}+\delta N_{11} & \cdots & N_{1n}+\delta N_{1n} \\ \vdots & \ddots & \vdots \\ N_{n1}+\delta N_{n1} & \cdots & N_{nn}+\delta N_{nn} \end{pmatrix}$$

Step 1:

$H^{'}$

$N^{'}$

Step 2: Solve generalized eigenvalue problem $H^{'}\lambda^{'} = EN^{'}\lambda^{'}$

Step 3: Repeat process *n* times to get a set of *n* eigenvalues $\longrightarrow$ $E \pm \delta E$

Trimmed Sampling part 1

$$\begin{pmatrix} H_{11}+\delta H_{11} & \cdots & H_{1n}+\delta H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1}+\delta H_{n1} & \cdots & H_{nn}+\delta H_{nn} \end{pmatrix} \qquad \begin{pmatrix} N_{11}+\delta N_{11} & \cdots & N_{1n}+\delta N_{1n} \\ \vdots & \ddots & \vdots \\ N_{n1}+\delta N_{n1} & \cdots & N_{nn}+\delta N_{nn} \end{pmatrix}$$

Step 1: Sample → H' ; Sample → N'

Step 2: Check if norm matrix is positive definite. If not reject this sample and go to step 1

Step 3: Solve generalized eigenvalue problem $H'\lambda' = EN'\lambda'$

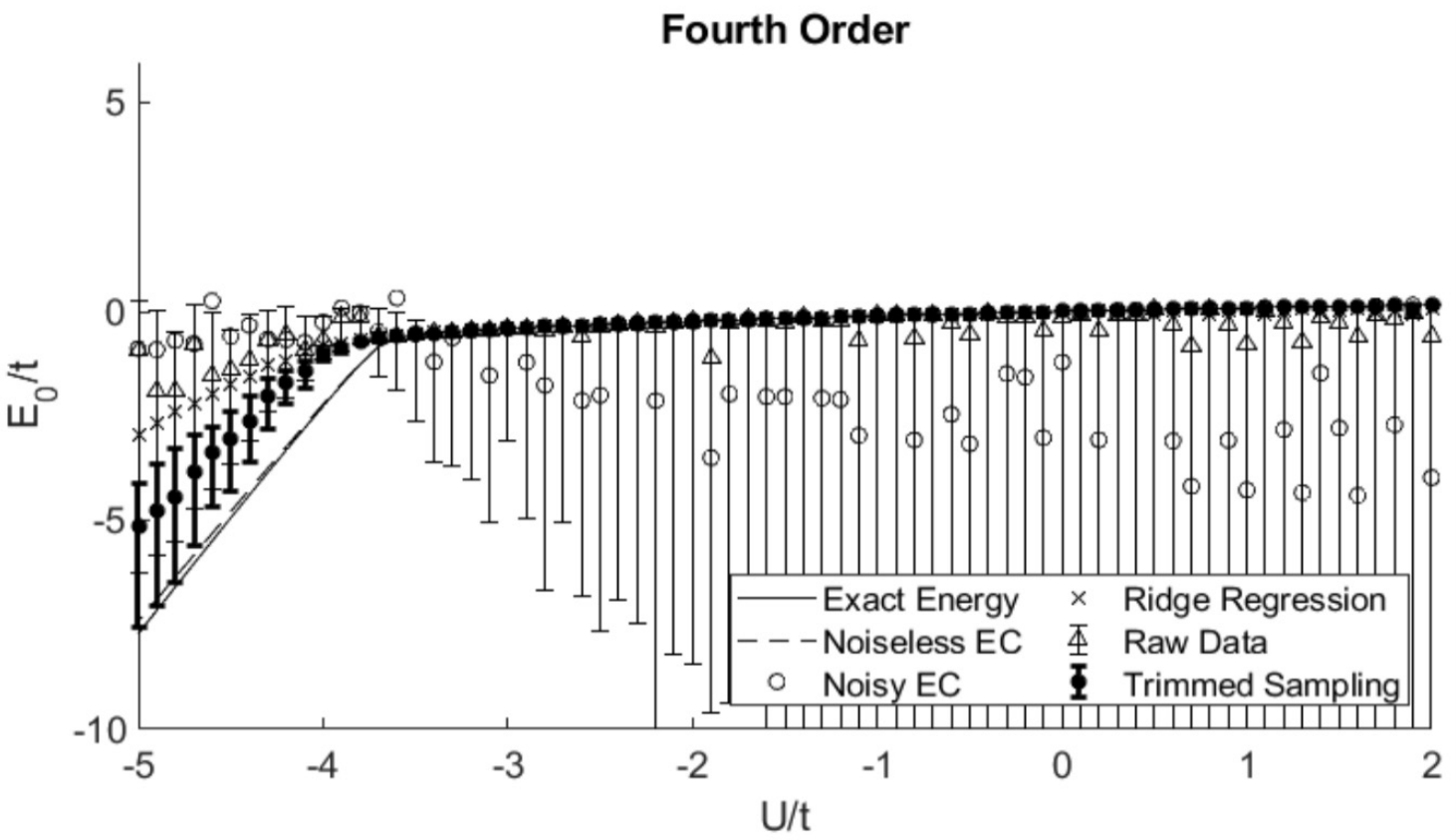
Step 4: Repeat process *n* times to get a set of *n* eigenvalues → $E \pm \delta E$

Trimmed sampling part 2

- Optional step: After sampling and solving for generalized eigenvalue E_n, check convergence of E_n as function of n, where n is dimension of $H_{n\times n}$ [$H_{n\times n}\psi = E_n N_{n\times n}\,\psi$]
- $C_n = \dfrac{E_n - E_{n-1}}{E_{n-1} - E_{n-2}}$ $\qquad C_{max} = \mathrm{Max}\{C_n\}$
- Assign weight to this sample $f = \exp(-\,C_{max}/\,C_{parameter})$
- From final weighted distribution get $E \pm \delta E$
- Penalize cases where eigenvalues are converging very slowly.
- Helps with stability of generalized eigenvalue solver.

Trimmed sampling example

- Bose-Hubbard model with 4 bosons on 4x4x4 lattice.
- Hamiltonian with hopping term proportional to t, a contact interaction proportional to U, and chemical potential $\mu = 6t$.
- Eigenvector continuation (EC) also solves a generalized eigenvalue problem.



Results (Preliminary)

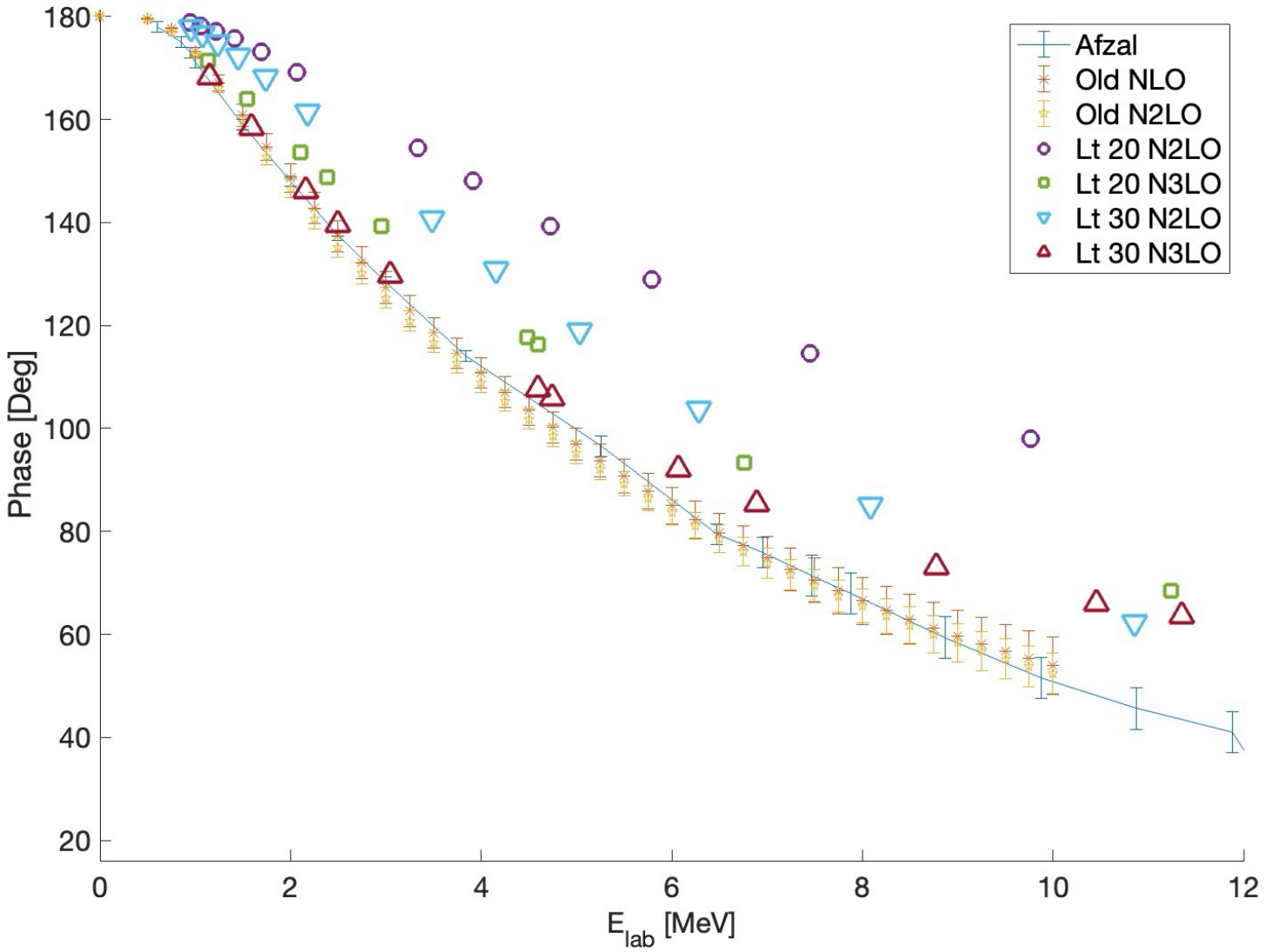
E_{APM} (Lt = 20, MeV)

	N2LO	N2LO with TS	N3LO	N3LO with TS
E_{Direct}	-43.80 ± 0.03		-57.34 ± 0.05	
E_{APM}	-49.63 (+5.02 -22.01)	-44.46 (+0.16 -1.68)	-69.21 (+10.70 - 33.42)	-58.48 (+0.21 - 1.29)
E_{APM} (8x statistics)	-44.60 ± 0.04	-44.60 ± 0.04	-59.34 (+0.32 - 0.74)	-59.84 (+0.32 - 0.62)

E_{APM} (Lt = 30, MeV)

	N2LO	N2LO with TS	N3LO	N3LO with TS
E_{Direct}	-45.03 ± 0.04		-57.45 ± 0.05	
E_{APM}	-45.97 (+0.14 -6.81)	-45.90 (+0.09 -0.21)	-60.63 (+1.27 -19.46)	-59.45 (+0.16 -0.77)

Phase shifts



Summary and Outlook

- Adiabatic Projection Method is an Ab initio method for studying scattering problems. It reduces the A-body problem to two cluster system.
- Better phase shifts than Lüscher's formalism
- Computationally expensive because of noise sensitivity, but trimmed sampling promises to boost computations significantly.
- Alpha-alpha s-wave scattering, and then p and d-waves.

Thank you

Supplemental Materials

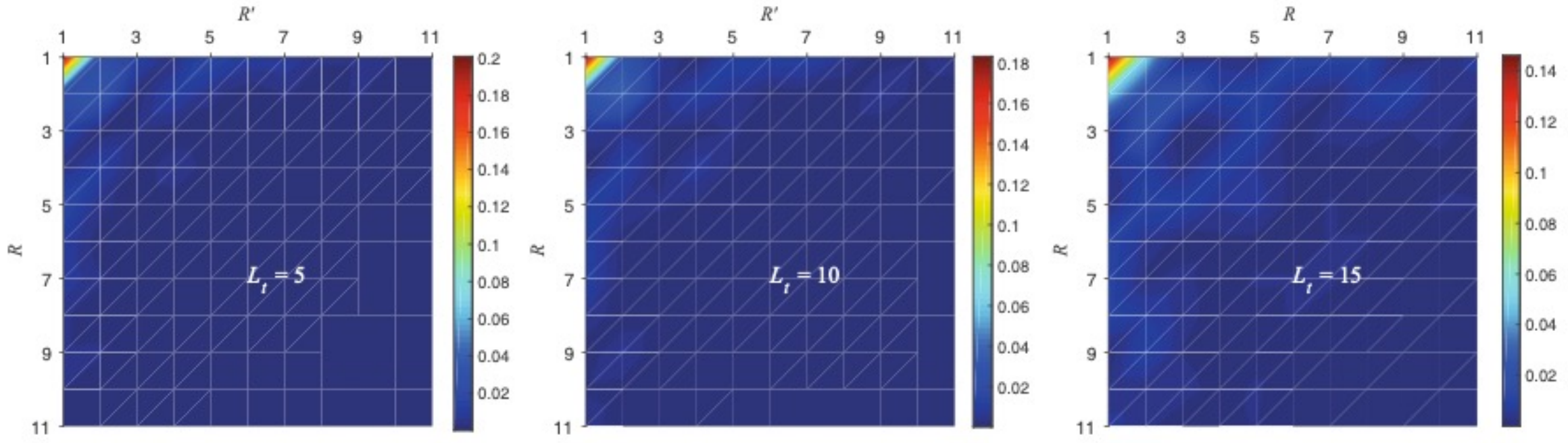


Fig. 3. The absolute difference between $[M^{\mathrm{a}}_{L_t}]^{\ell,\ell_z}_{R,R'}$ and $[M^{\mathrm{eff}}]^{\ell,\ell_z}_{R,R'}$ for the fermion-dimer system with the angular quantum numbers $\ell = 0$ and $\ell_z = 0$ and for the Euclidean time steps $L_t = 5$, 10, and 15, respectively. The absolute difference matrices are in lattice units. For $L_t = 15$ the magnitude of the absolute difference is equal to 18% of the corresponding value of the trivial radial adiabatic transfer matrix for the lattice site where the absolute difference is maximum. The physical distances for indices R, $R' = 1$, 3, 5, 7, 9, and 11 are 2.22, 3.20, 4.68, 6.16, 7.14, and 8.62 fm, respectively.

[1] S. Elhatisari, D. Lee, U-G. Meißner, G. Rupak, Eur. Phys. J. A 52, 174 (2016)

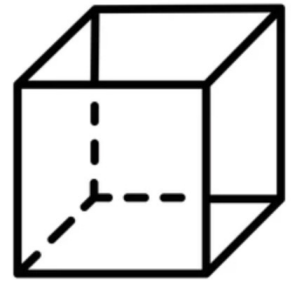
Lüscher formalism

- Lüscher formula [1] gives phase shifts from which we can determine E_0 and Γ.

$$\det\left[\,\cos\delta - \sin\delta\; F^{(FV)}\,\right] = 0$$

$$\overline{F}^{(FV)}_{l_1 m_1; l_2 m_2} = \frac{(-)^{m_2}}{\tilde{q}\;\pi^{3/2}}\sqrt{(2l_1+1)(2l_2+1)}\sum_{\bar{l}=|l_1-l_2|}^{|l_1+l_2|}\sum_{\overline{m}=-\bar{l}}^{\bar{l}}\frac{\sqrt{2\bar{l}+1}}{\tilde{q}^{\bar{l}}}\begin{pmatrix} l_1 & \bar{l} & l_2 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & \bar{l} & l_2 \\ -m_1 & -\overline{m} & m_2\end{pmatrix}\mathcal{Z}_{\bar{l},\overline{m}}(1;\tilde{q}^2)$$

$$\mathcal{Z}_{l,m}(s;\tilde{q}^2) = \sum_{\mathbf{n}}\frac{|\mathbf{n}|^l\; Y_{lm}(\Omega_{\mathbf{n}})}{\left[\;|\mathbf{n}|^2 - \tilde{q}^2\;\right]^s}$$



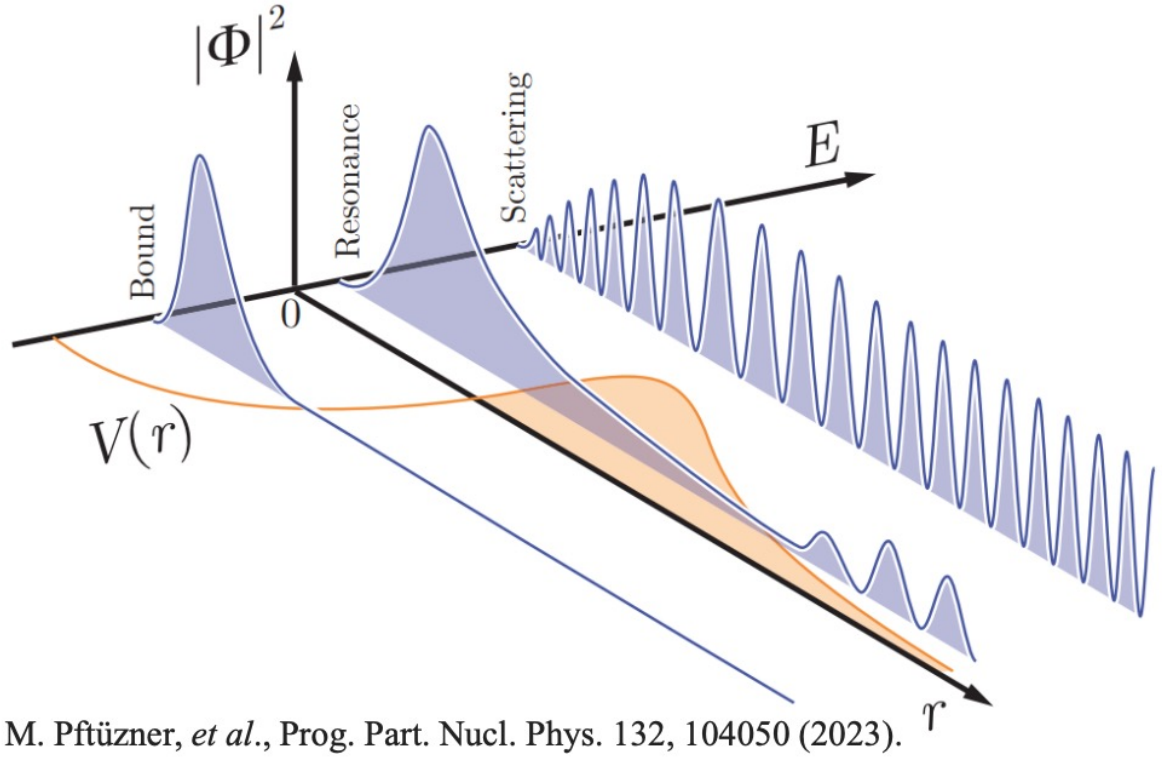
- Highly sensitive to noise.
- Additional diffculties with higher partial wave and coupled channel phase shifts.

[1] – M. Lüscher, Comms. Math. Phys. 105, 153 (1986)

Resonance in scattering problems

- Quasi-bound, finite lifetime.
- Poles of the S-matrix in the complex plane ($E_0 - i\Gamma/2$).
- A peak in cross section in experiments.
- Common resonance behavior - Breit-Wigner distribution.

$$E \propto \frac{\Gamma}{(E-E_0)^2+\Gamma^2}$$



M. Pftüzner, *et al.*, Prog. Part. Nucl. Phys. 132, 104050 (2023).

Survival amplitude

- Survival amplitude of a state

$$f(t) = \langle\psi|e^{-iHt}|\psi\rangle$$

$$|\psi\rangle = \int dE\ a(E)|E\rangle \quad \Rightarrow \quad f(t) = \int dE\ |a(E)|^2\ e^{-iEt}$$

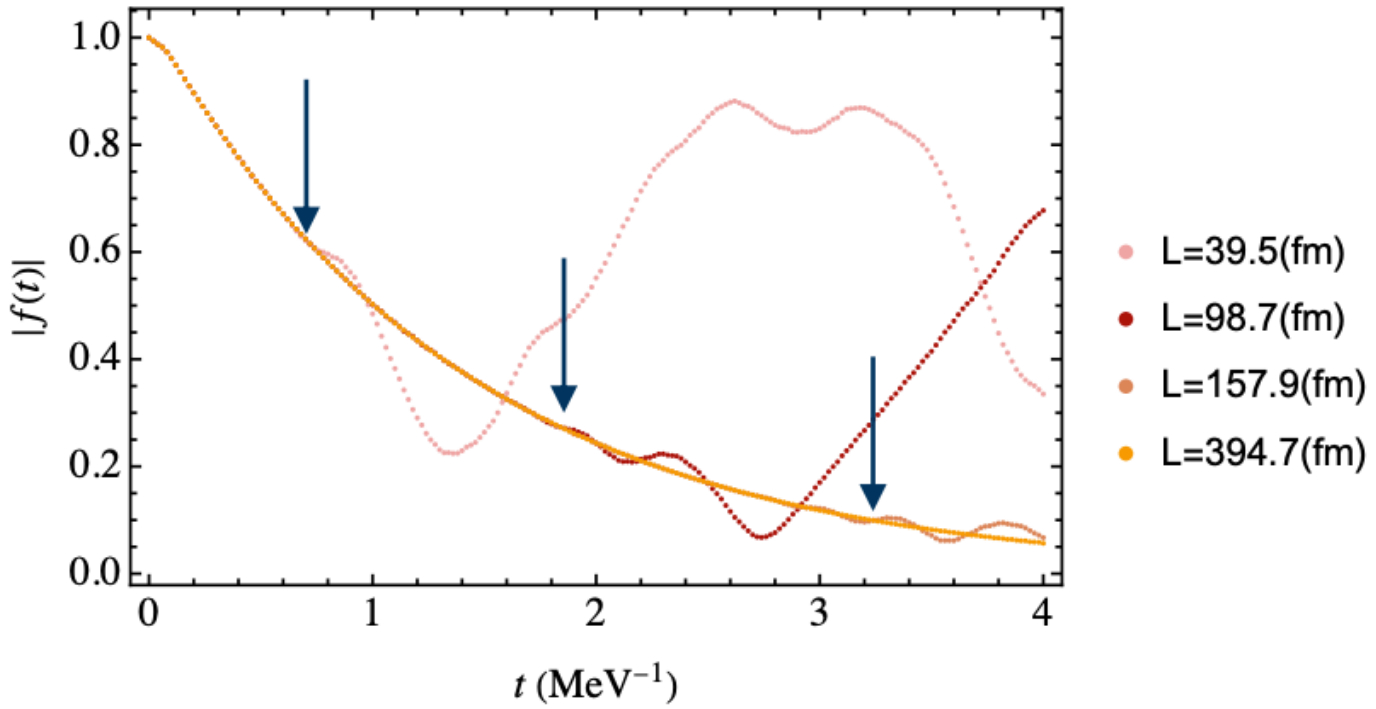
- For resonant state with Breit-Wigner parameterization

$$a(E) \propto \frac{1}{E\ -\ E_0 + i\Gamma/2} \quad \Rightarrow \quad f(t) \propto\ e^{-iE_0 t\ -\ \Gamma t/2}$$

- Choose compact initial state with good overlap with the resonant state.

Resonance from NLEFT

- Can only do Eucledian time evolution
  - $\exp(-H\tau)$ vs $\exp(-iHt)$
- Computationally limited to small box sizes.
  - Finite volume effect
- Eucledian time evolution means eigennstates are decaying. The fit results depends on region where we fit.



Wavefunction Matching