Lattice QCD and effective field theories

Akaki Rusetsky, University of Bonn

GGSWBS’14, Tbilisi, 8 July 2014

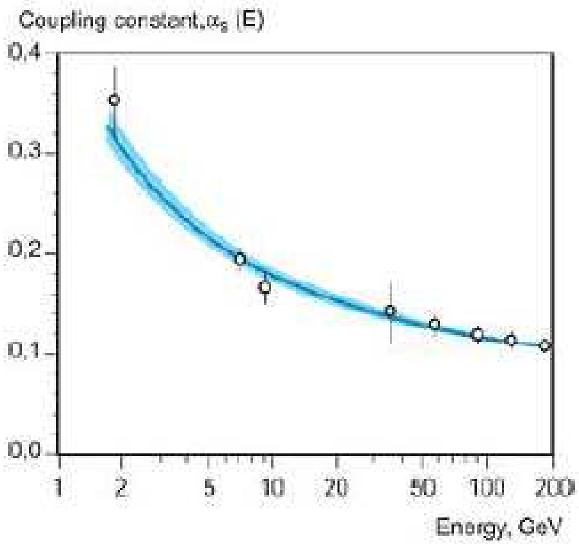
Plan

- Introduction
  - Why lattice?
  - Why effective field theories?
- Scattering observables from lattice simulations
- Lüscher equation and resonances
- The role of the boundary conditions
- Testing the nature of the observed states on the lattice
- Further applications: scattering amplitudes, form factors, three and more particles, . . .
- Conclusions, outlook

QCD on the lattice

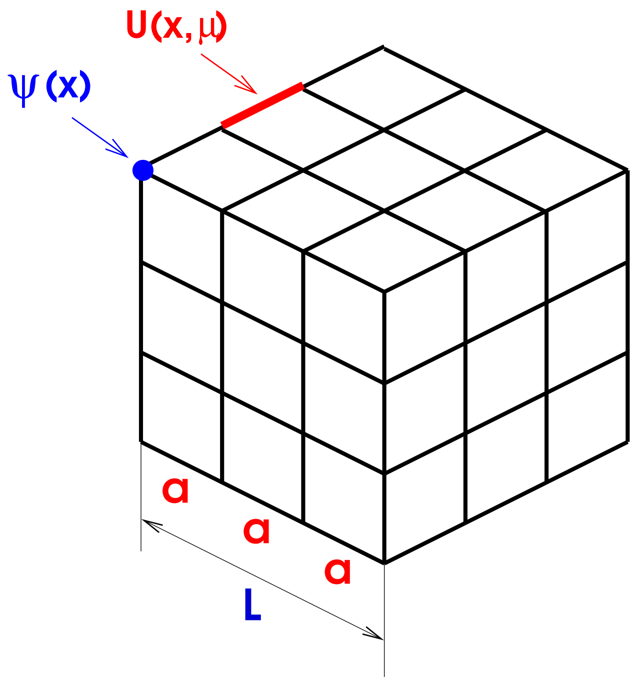
$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} + \bar{\psi}(i\gamma^\mu(\partial_\mu - igT^a G^a_\mu) - \mathcal{M})\psi$$

$$F^a_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - igf^{abc} G^b_\mu G^c_\nu \,, \qquad \mathcal{M} = \text{diag}\,(m_u, m_d, \cdots)$$



Non-perturbative at low energies:

- Confinement
- Spontaneous chiral symmetry breaking



$\hookrightarrow$ QCD on the lattice

The energy spectrum of the lattice Hamiltonian

- The energy spectrum on the lattice is discrete (finite volume)
- The behavior of the two-point function in the Euclidean space at large time separation

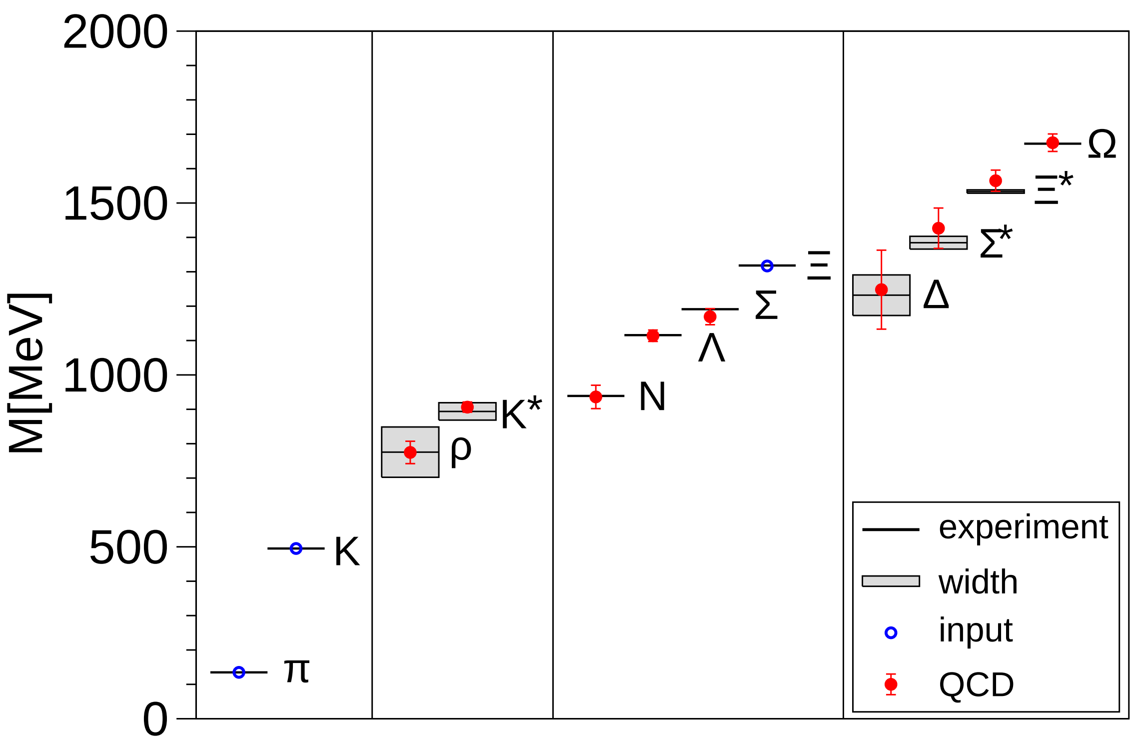
$$C(t) = \langle 0|\mathcal{O}(t)\mathcal{O}^\dagger(0)|0\rangle = \int dU\, d\psi d\bar\psi\, \mathrm{e}^{-S_{QCD}(U,\psi,\bar\psi)}\, \mathcal{O}(t)\mathcal{O}^\dagger(0)$$

  Determines the lowest energy level with quantum numbers of $\mathcal{O}$:

$$C(t) = \sum_n |\langle 0|\mathcal{O}(0)|n\rangle|^2 \mathrm{e}^{-E_n t} \quad \to \quad |\langle 0|\mathcal{O}(0)|1\rangle|^2\, \mathrm{e}^{-E_1 t} + \cdots$$

- Excited states $E_2,\ E_3,\ldots$ can be also extracted by using, e.g., the variational method

The low-energy spectrum of QCD



Recent results on the meson and baryon spectrum in QCD,
S. Dürr *et al.*, Science 322 (2008) 1224

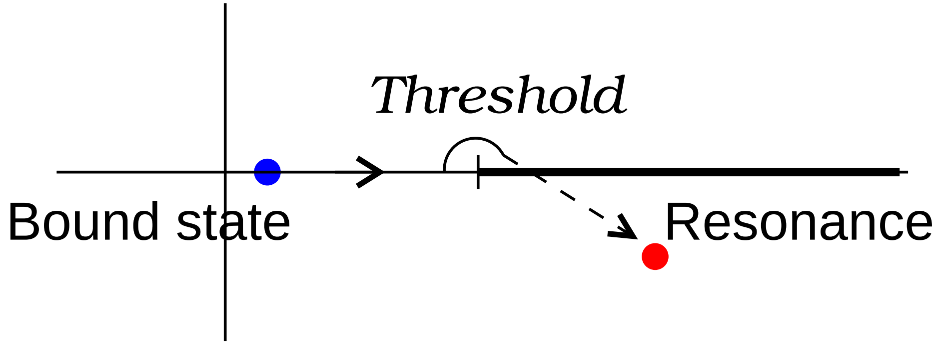
Multiparticle scattering states?

Resonances?

Resonances

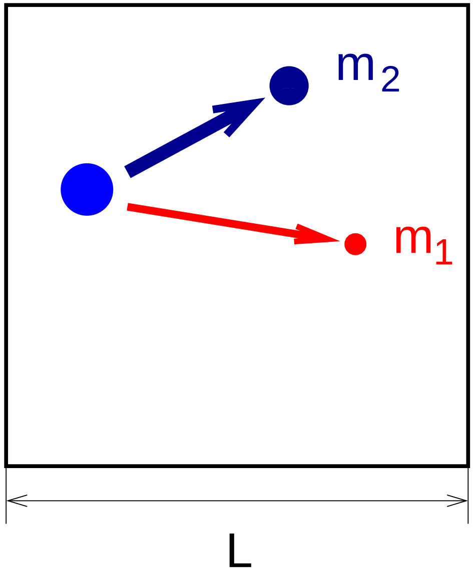
- Most of particles in QCD are resonances: $\Delta,\ \rho,\ \ldots$
- Resonances are characterized by their mass, their lifetime, ...
- These are the *intrinsic* properties of a resonance that should not depend neither on a particular *experiment* nor a particular *theoretical model* which is used to describe the data

$\hookrightarrow$ Resonances correspond to S-matrix poles on the unphysical Riemann sheets



How does one extract a resonance mass and width on the lattice?

Does one observe a resonance on the lattice?

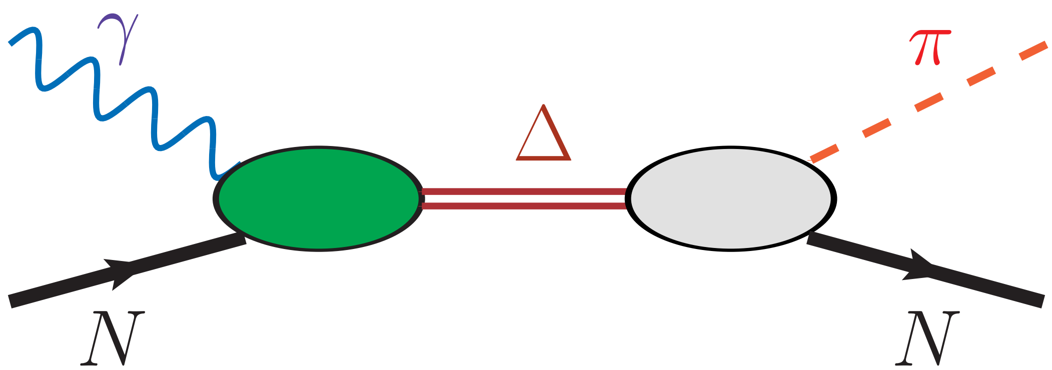


$$C(t) = \langle 0|\mathcal{O}(t)\mathcal{O}^\dagger(0)|0\rangle = \sum_n |\langle 0|\mathcal{O}(0)|n\rangle|^2 \mathrm{e}^{-E_n t}$$

$$E_n(L) = m_1 + m_2 + \Delta E_n(L)\,, \qquad \lim_{L\to\infty} \Delta E_n(L) = 0$$

If you wait sufficiently long, a resonance on the lattice decays . . .

Resonance matrix elements in the continuum



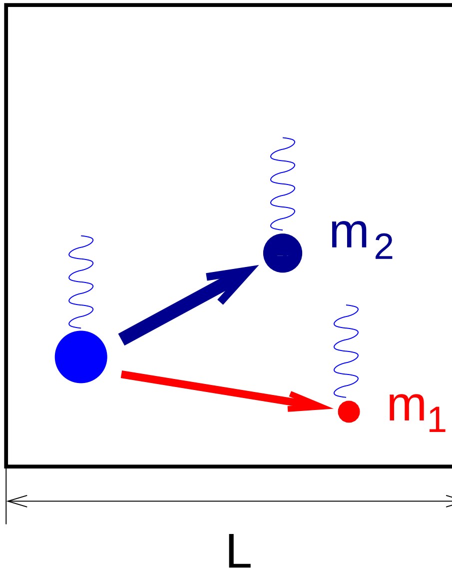
In the vicinity of the resonance pole...

$$\Rightarrow \quad i\int d^4x\, e^{iPx}\, \langle 0|T\Delta(x)\bar\Delta(0)|0\rangle \to \frac{Z_R}{s_R-P^2}+\cdots$$

$$\Rightarrow \quad i^2\int d^4x d^4y\, e^{iPx-iQy}\, \langle 0|T\Delta(x)J(0)\bar N(y)|0\rangle$$

$$\to \frac{Z_R^{1/2}}{s_R-P^2}\,\langle\Delta|J(0)|N\rangle\,\frac{Z_N^{1/2}}{m_N^2-Q^2}+\cdots$$

Resonance matrix elements on the lattice



$$\langle \Delta | J_\mu(0) | \Delta \rangle = \langle N | J_\mu(0) | N \rangle + \langle \pi | J_\mu(0) | \pi \rangle + \Delta F_\mu(L)$$

$$\lim_{L \to \infty} \Delta F_\mu(L) = 0$$

The nature of the resonances

Is a given resonance . . .

- a “standard” bound state of quarks: $q\bar{q}$ or qqq ?

  Beware: resonances, not stable quark model bound states!

- a hadronic molecule (a bound state of hadrons)?
- an “exotic” state: glueball, tetraquark, pentaquark, etc?

$\hookrightarrow$ What are the relevant criteria?

Lüscher's approach: resonances from the lattice data

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

- Lattice simulations are always done at:

  Finite volume (box size L)

  Finite lattice spacing a

Lüscher's approach: $R^{-1}L \simeq ML \gg 1\,, \qquad a \to 0$

R: the range of interaction

- Momenta are small: $p \simeq 2\pi/L \ll$ the lightest mass
- Finite-volume corrections to the energy levels are only power-suppressed in L
- Studying the dependence of the energy levels on L gives the scattering phase in the infinite volume $\Rightarrow$ **Resonances**

Why effective field theories?

Most general effective Lagrangian: all terms with assumed symmetry

$\hookrightarrow$ Most general S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and symmetries (Weinberg, 1979)

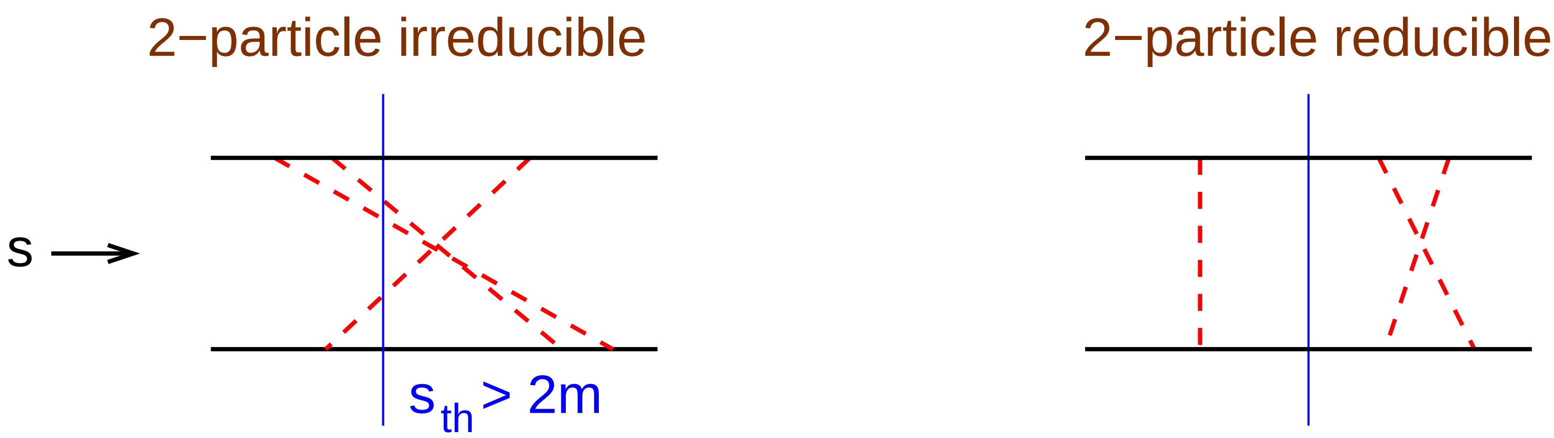
QCD $\longrightarrow$ ChPT $(p \sim M_\pi)$ $\longrightarrow$ non-rel. EFT $(p \ll M_\pi)$

- Lattice QCD $\longrightarrow$ EFT in a finite volume
- Short-distance behavior not changed: the same Lagrangian

$\hookrightarrow$ A bridge between the measured spectrum and scattering sector:

Multichannel resonances
Twisted boundary conditions
Lüscher-Lellouch formula, timelike pion formfactor
Resonance formfactors
Few-body systems

Relativistic vs non-relativistic EFT



Exponentially suppressed for $s < s_{th}$

- Well below inelastic threshold, finite-volume corrections to the 2-particle irreducible diagrams are exponentially suppressed $\sim \exp(-\Delta E \cdot L)$
- 2-particle irreducible diagrams $\longrightarrow$ L-independent local vertices

$\hookrightarrow$ Particle number is conserved in intermediate states

Covariant NREFT in the infinite volume: $p \sim L^{-1} \ll M_\pi$

G. Colangelo, J. Gasser, B. Kubis and AR, PLB 638 (2006) 187

J. Gasser, B. Kubis and AR, NPB 850 (2011) 96

Non-relativistic Lagrangian: no particle criation/annihilation

$$\mathcal{L} = \sum_i \Phi_i^\dagger (2W_i)(i\partial_t - W_i)\Phi_i + C_0 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2$$
$$+ \quad C_1 \Big((\Phi_1^\dagger)^\mu (\Phi_2^\dagger)_\mu \Phi_1 \Phi_2 - M_1 M_2 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 + \text{h.c.} \Big) + \cdots$$

$$W_i = \sqrt{M_i^2 - \triangle}, \qquad \Phi_i(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2W_i(\mathbf{p})}\, e^{-ipx} a(\mathbf{p})$$

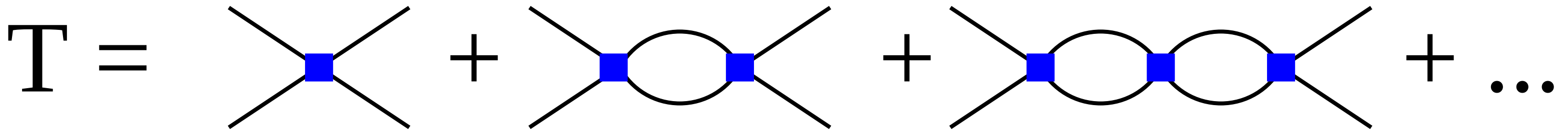
Relativistic dispersion law in the propagator:

$$D_i(p) = \frac{1}{2W_i(\mathbf{p})}\,\frac{1}{W_i(\mathbf{p}) - p_0 - i0}$$

Lippmann-Schwinger equation

- Expand $W_i(\mathbf{p}) = M_i + \mathbf{p}^2/(2M_i) + \cdots$ in all Feynman integrands, integrate in the dimensional regularization and sum up again

$$\hookrightarrow \text{loop} = \frac{ip}{8\pi\sqrt{s}}\,, \qquad p = \frac{\lambda^{1/2}(s, M_1^2, M_2^2)}{2\sqrt{s}}$$



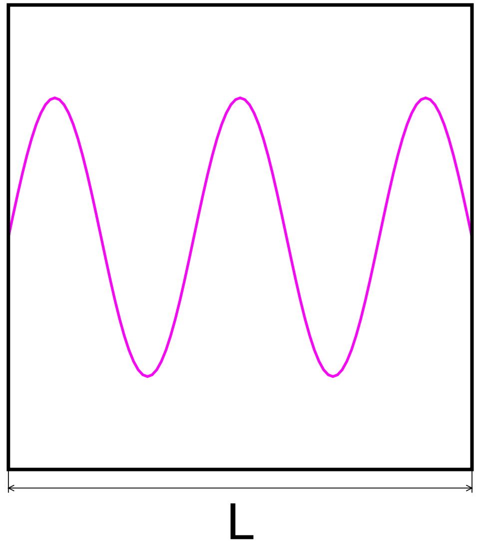
$\hookrightarrow$ Scattering amplitude is Lorentz-invariant:

$$T_l = \frac{8\pi\sqrt{s}}{p\cot\delta_l(p) - ip}$$

- Important in nonrest frames (formfactors, 3-body scattering)

Quantization of momenta

Periodic boundary conditions for the w.f.: $\psi(\mathbf{x}+L\mathbf{e}_i)=\psi(\mathbf{x})$



Standing waves: $\psi_n(\mathbf{x})=e^{i\mathbf{p}_n\mathbf{x}}\,,\qquad \mathbf{p}_n=\frac{2\pi}{L}\,\mathbf{n}\,,\quad \mathbf{n}\in\mathbb{Z}^3$

Feynman diagrams:

$$\int\frac{d^3\mathbf{p}}{(2\pi)^3}\,F(\mathbf{p})\to\frac{1}{L^3}\sum_{\mathbf{n}}F(\mathbf{p}_n)$$

Covariant NREFT in a finite volume

Loops modified, Lüscher's zeta-function emerges:

$$\text{loop} = \frac{ip}{8\pi\sqrt{s}} \to \frac{Z_{00}}{4\pi^{3/2}L\sqrt{s}} \qquad \text{(S-wave)}$$

$$Z_{lm}(1,q^2) = \lim_{s\to 1} \sum_{\mathbf{n}\in\mathbb{Z}^3} \frac{\mathcal{Y}_{lm}(\mathbf{n})}{(\mathbf{n}^2-q^2)^s}\,, \qquad q = \frac{pL}{2\pi}$$

Poles in the LS equation = spectrum of the Hamiltionan

(see S. R. Beane *et al.*, NPA 747 (2005) 55)

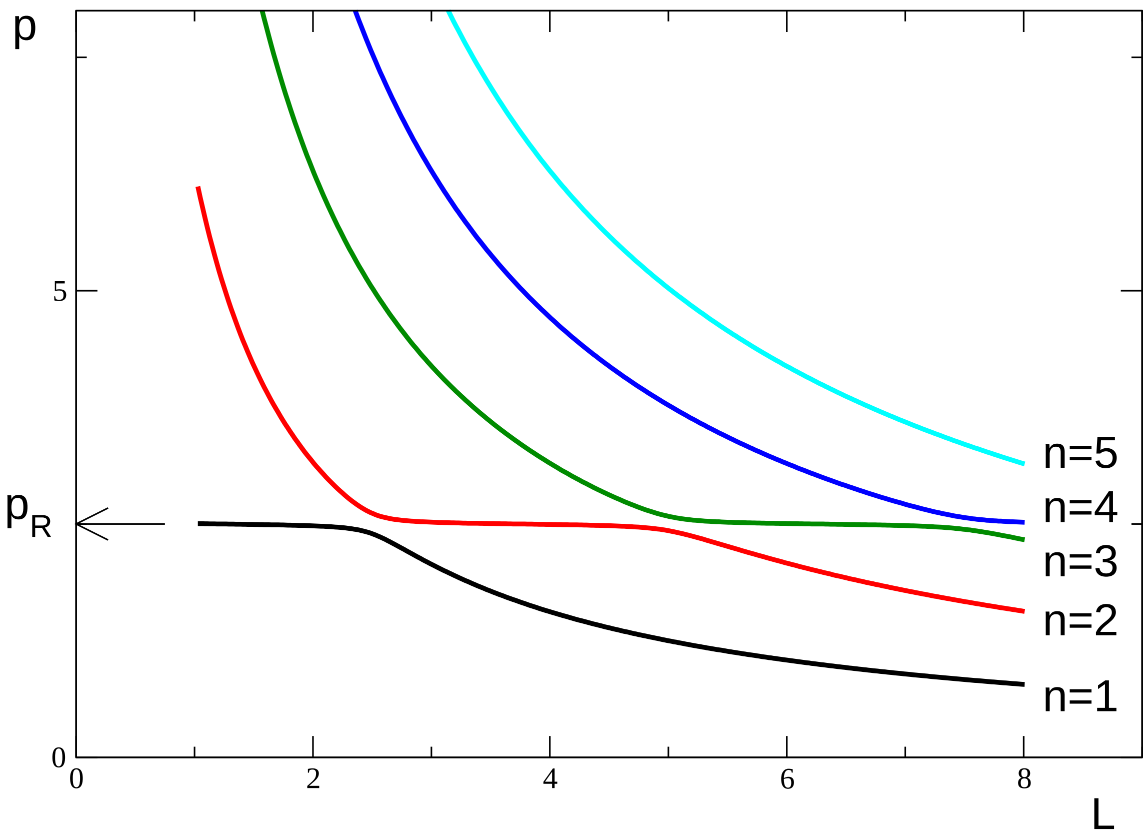
$\hookrightarrow$ Lüscher equation:

$$\det\left(\delta_{ll'}\delta_{mm'} - \tan\delta_l(s)\mathcal{M}_{lm,l'm'}\right) = 0$$

- $\mathcal{M}_{lm,l'm'}$ is a linear combination of Z_{lm} [partial-wave mixing]

Energy levels in the presence of a narrow resonance

⇒ Lüscher formula predicts an irregular behavior of the levels in the vicinity of a narrow resonance: *avoided level crossing*



Resonances are not described by a single energy level!

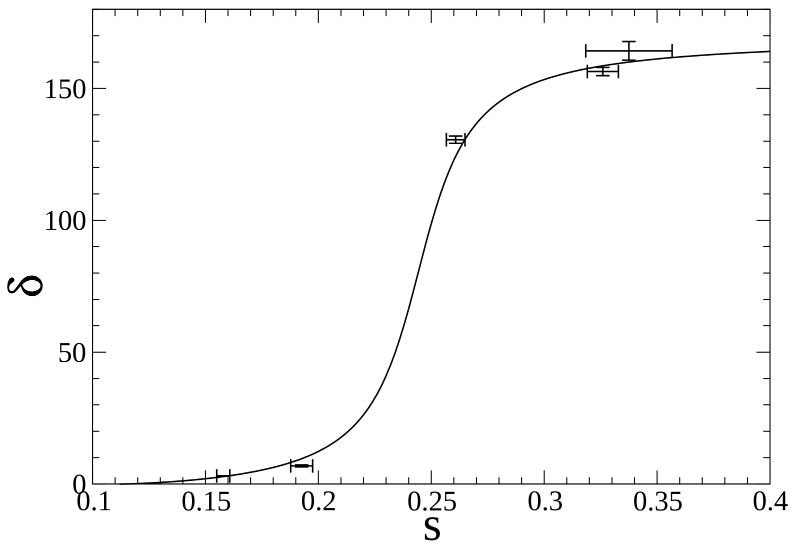
Where are the resonance poles?

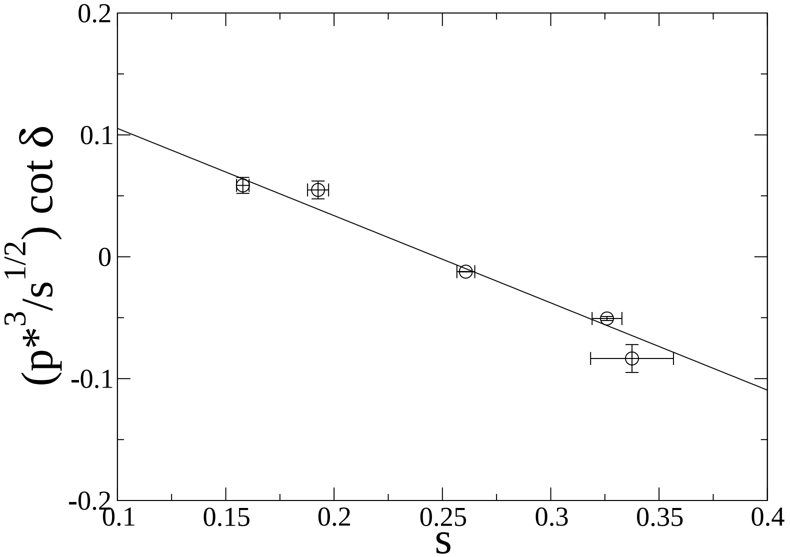
Assume effective range expansion:

$$p \cot\,\delta(p) = A_0 + A_1 p^2 + \cdots\,, \qquad \cot\,\delta(p_R) = -i \quad \surd$$

⇒ $A_0, A_1, \cdots$ are measured on the lattice

⇒ Resonance pole p_R in the complex momentum plane





Phase shift for the ρ-meson: S. Prelovsek *et al.*, arXiv:1111.0409

Example 1: Lüscher equation with coupled channels

M. Lage, U.-G. Meißner and AR, PLB 681 (2009) 439

V. Bernard, M. Lage, U.-G. Meißner and AR, JHEP 1101 (2011) 019

C. Liu, X. Feng and S. He, JHEP 0507 (2005) 011; Int. J. Mod. Phys. A 21 (2006) 847

M. T. Hansen and S. R. Sharpe, PRD 86 (2012) 016007

R. A. Briceno and Z. Davoudi, arXiv:1204.1110 [hep-lat]

N. Li and C. Liu, PRD 87 (2013) 014502

P. Guo, J. Dudek, R. Edwards and A. P. Szczepaniak, PRD 88 (2013) 014501

Two-channel equation:

$$\begin{aligned} T_{11} &= K_{11} + K_{11} i p_1 T_{11} + K_{12} i p_2 T_{21} \\ T_{21} &= K_{21} + K_{21} i p_1 T_{11} + K_{22} i p_2 T_{21} \end{aligned}$$

Resonance pole(s) are determined from the secular equation:

$$1 - ip_1 K_{11} - ip_2 K_{22} - p_1 p_2 (K_{11} K_{22} - K_{12}^2) = 0$$

Multi-channel system in a finite volume

⇒ Multi-channel Lüscher equation: $ip_i \to \dfrac{2}{\sqrt{\pi}L} Z_{00}(1, q_i^2)$

⇒ Find $K_{ij}(s)$ from the lattice data: one equation for K_{11}, K_{12}, K_{22}

⇒ A parameterization of the K-matrix is needed; example: the effective-range expansion

⇒ Alternative: use dynamical input from unitary ChPT at lowest order, approximate potentials by the polynomials in s

M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139

$$V_{ij}(s) = V_{ij}^{(0)} + V_{ij}^{(1)}(s - s_t) + \cdots$$

⇒ Find the position of the pole(s) from the secular equation

Further developments, see e.g.:

M. Döring *et al.*, EPJA 47 (2011) 163 [$\bar{K}N - \pi\Sigma$ scattering]
M. Döring and U.-G. Meißner, JHEP 1201 (2012) 009 [πK scattering]
M. Döring, M. Mai and U.-G. Meißner, PLB 722 (2013) 185 [πN scattering]

Twisted boundary conditions

P.F. Bedaque, PLB 593 (2004) 82
G.M. de Diviitis, R. Petronzio and N. Tantalo, PLB 595 (2004) 408
G.M. de Diviitis and N. Tantalo, hep-lat/0409154
C.T. Sachrajda and G. Villadoro, PLB 609 (2005) 73

The twisting angle: $\boldsymbol{\theta}_i \in (0, 2\pi)$ $\Rightarrow$ quark momentum: $\mathbf{p}_i = \boldsymbol{\theta}_i / L$

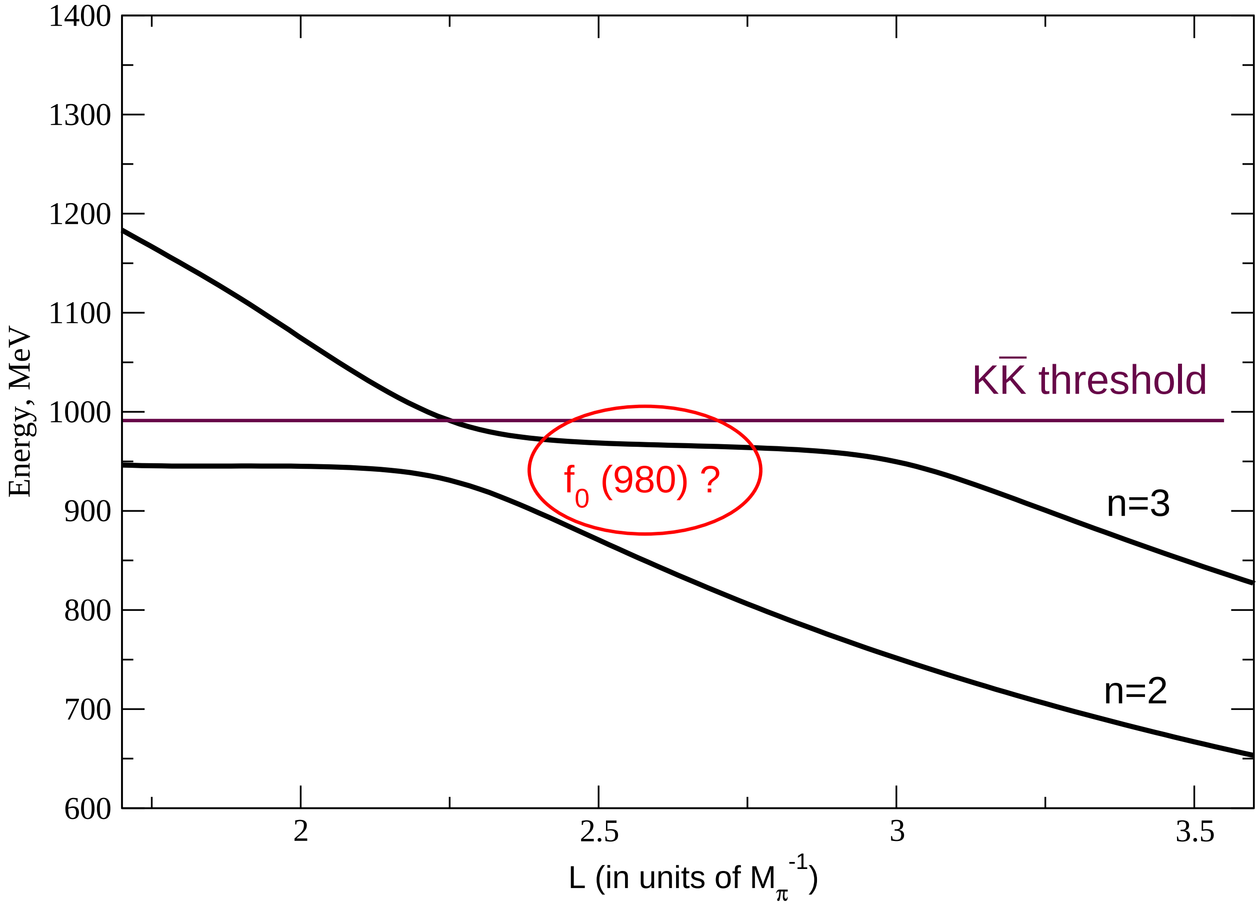
$$\psi_i(\mathbf{x} + \mathbf{n}L) = e^{i\mathbf{n}\boldsymbol{\theta}_i} \psi_i(\mathbf{x}) \,, \qquad i = u, d, s$$

Modification of the zeta-function (take CM frame for simplicity)

$$Z_{00}(1; q^2) \to Z_{00}^{\theta}(1, q^2) = \frac{1}{\sqrt{4\pi}} \lim_{s \to 1} \sum_{\mathbf{n} \in Z^3} \frac{1}{((\mathbf{n} - \boldsymbol{\theta}/2\pi)^2 - q^2)^s}$$

$\Rightarrow$ Accurate extraction of the resonance parameters from data!

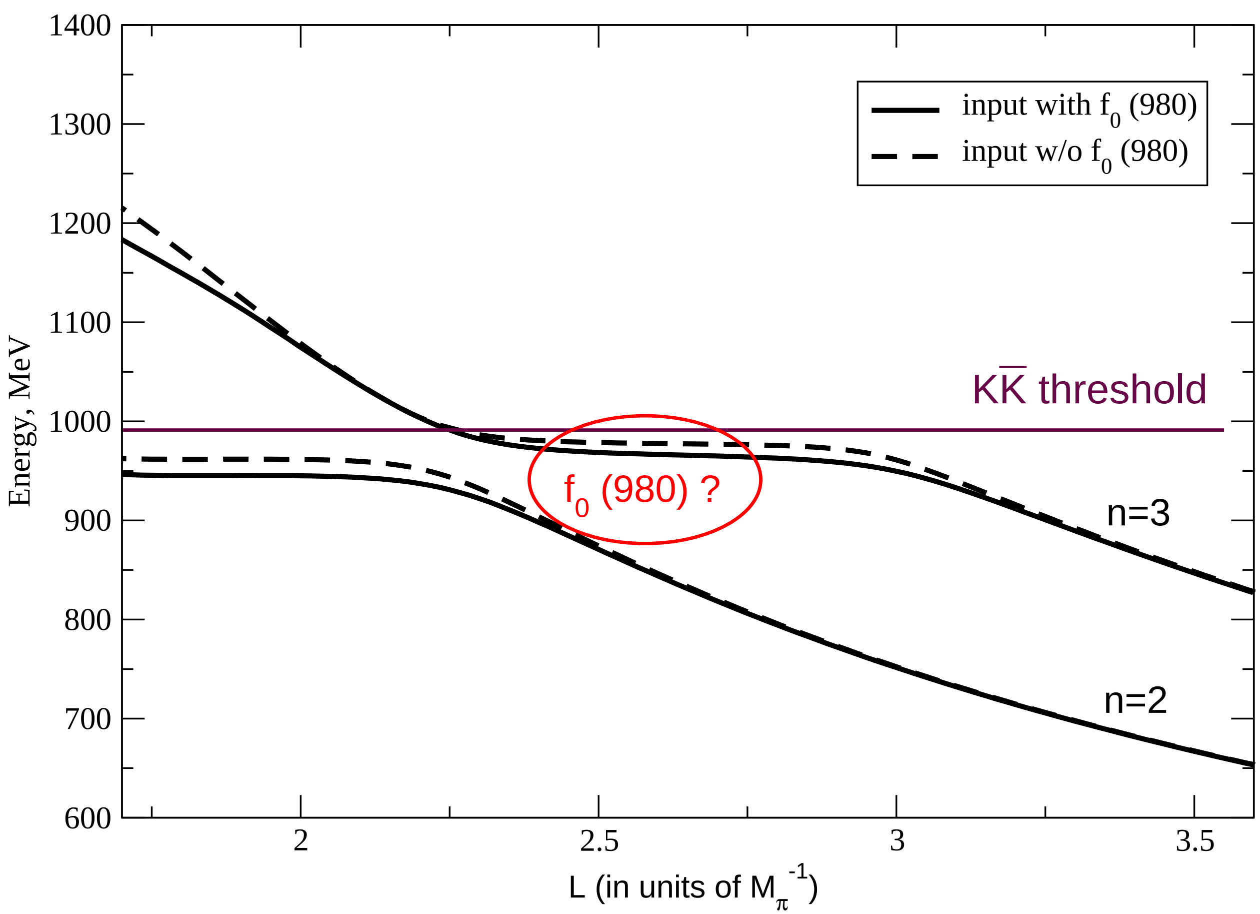
“Masking” resonances by a threshold: $f_0(980)$



Using Unitarized Chiral Perturbation Theory to produce energy levels

M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139

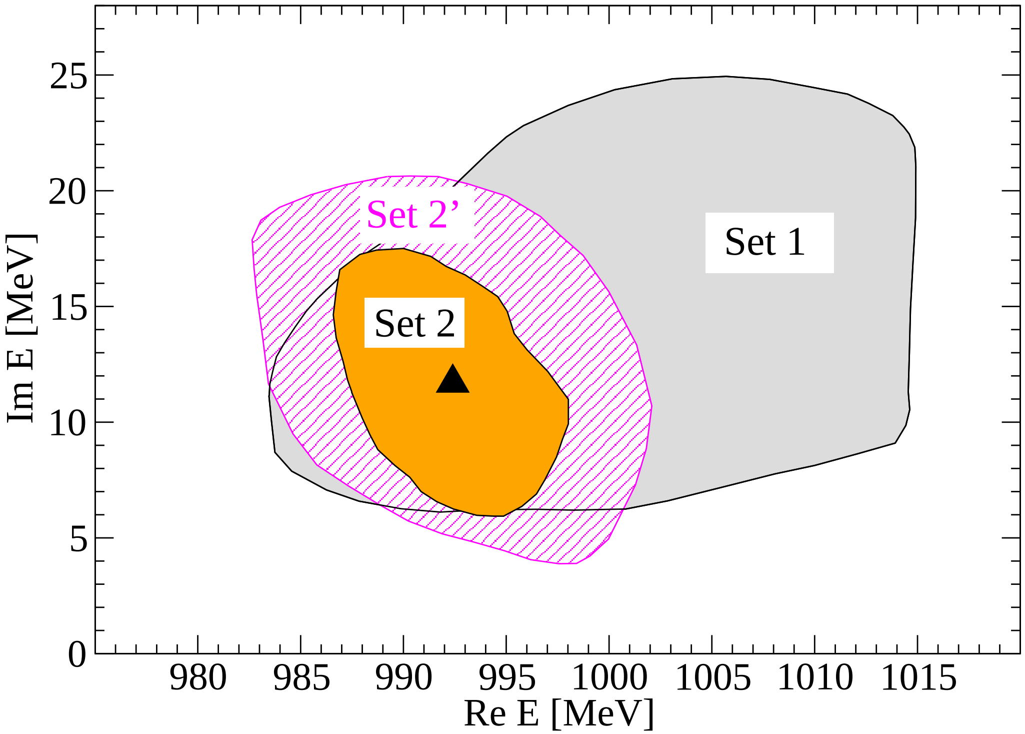
Changing the input . . .



- Weaker coupling to the $K\bar{K}$ channel, $f_0(980)$ disappears
- Avoided level crossing still occurs at the same place

Extraction of the $f_0(980)$ pole position, CM frame

M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139



13 (synthetic) data points in each set:

Set 1: Energy levels 2+3, periodic b.c.

Set 2: Energy level 2, periodic + antiperiodic b.c. preferrable!

Set 2': Set 2 + statistical error

Example 2: the nature of the $D_{s0}^*(2317)$ meson

D. Agadjanov, F.-K. Guo, G. Rios and AR, in preparation

$D_{s0}^*(2317)$ lies below the DK threshold $\Longrightarrow$ stable!

- Tight $\bar{c}s$ quark compound with $J^P = 0^+$?
- DK bound state?

Weinberg's compositeness condition: w.f. renormalization constant

Dressed propagator:

$$D(s) = \frac{1}{M_0^2 - s + g_0^2 G(s)} \to \frac{Z}{M^2 - s}$$

$$Z = \frac{1}{1 - g_0^2 G'(M^2)} \doteq 1 + g^2 G'(M^2)$$

$$G(s) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{2w_1 w_2} \frac{w_1 + w_2}{s - (w_1 + w_2)^2}$$

Compositeness condition

For shallow bound states, Z can be measured:

$$\text{scatt. length} = \frac{2R(1-Z)}{2-Z}\,, \qquad \text{eff. range} = \frac{RZ}{1-Z}$$

- $0 < Z < 1$
- $Z \simeq 1$: a tight quark compound
- $Z \simeq 0$: a large moleqular component

$\hookrightarrow$ For shallow bound states: a universal criterion to reveal the molecular nature of a state

Is it possible to formulate the compositeness condition on the lattice?

Compositness condition on the lattice

The position of the bound-state pole in the infinite volume:

$$T = V + VGT \quad \Rightarrow \quad V^{-1}(M^2) - G(M^2) = 0$$

The shift of the pole in a finite volume $M^2 \to M_L^2$:

$$V^{-1}(M_L^2) - G_L(M_L^2) = 0 \quad \Rightarrow \quad T^{-1}(M_L^2) - (G_L(M_L^2) - G(M_L^2)) = 0$$

$$\Delta M_L = -\frac{6\kappa}{\mu L} \, \frac{1}{1 - 2\kappa(p\cot\delta(p))'} \, e^{-\kappa L} + \cdots = -\frac{6\kappa(1-Z)}{\mu L} \, e^{-\kappa L} + \cdots$$

$\hookrightarrow$ Z can be measured on the lattice from the measurement of the bound-state energy at different volumes!

A. Martinez Torrez *et al.*, PRD 85 (2012) 014027; T. Sekihara *et al.*, PRC 87 (2013) 045202

Using twisted boundary conditions

- Performing the measurement of the energy spectrum at different volumes is a quite expensive enterprise
- A cheap alternative: using twisted boundary conditions
- Example: CM frame, twist along the thrid axis $\boldsymbol{\theta} = (0, 0, \theta)$

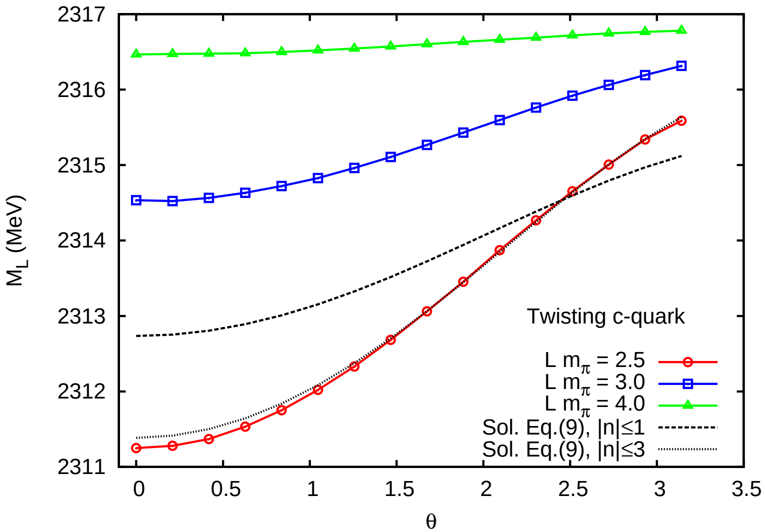
$$G_L^\theta - G = -\frac{1}{8\pi ML}\left\{(4 + 2\cos\theta)e^{-\kappa r} + \sqrt{2}(2 + 4\cos\theta)e^{-\sqrt{2}\kappa r} + \cdots\right\}$$

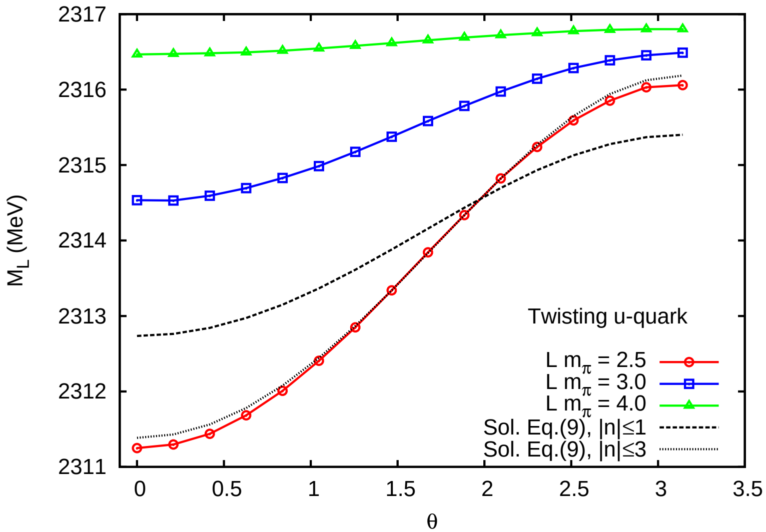
- Can be easily generalized for moving frames, different twist

Instead of performing simulations at different volumes, study the dependence of the spectrum at different values of the twisting angle θ

Dependence of the spectrum on the twisting angle

Synthetic data produced in the model F.-K. Guo *et al.*, PLB 641 (2006) 278





Sensitivity with respect to the variation of θ and L comparable!

$\hookrightarrow$ Using (partially) twisted boundary conditions represents a preferred strategy in the study of $D^*_{s0}(2317)$

Example 3. Measuring the $\Delta N\gamma^*$ form factor on the lattice

A. Agadjanov, V. Bernard, U.-G. Meißner and AR, arXiv:1405.3476 [hep-lat]

Form factor can be measured, assuming Δ to be a stable particle

C. Alexandrou *et al.,* PRD 79 (2009) 14507; arXiv:1108.4112; PRD 83 (2011) 014501

How the formalism is generalized in case of an unstable Δ?

- Which quantities should be measured on the lattice?
- How does one perform the infinite-volume limit in the form factors?
- How does one calculate the photoproduction amplitude?
- How does one perform the analytic continuation to the resonance pole?
- How does one project out different form factors in case of the unstable Δ?

$\hookrightarrow$ Use EFT approach in a finite volume

Kinematics

$$\Delta(t) = \sum_{\mathbf{x}} \Delta(t,\mathbf{x}) \quad \text{(CM frame)}\ , \qquad N(t) = \sum_{\mathbf{x}} e^{-i\mathbf{Q}\mathbf{x}} N(t,\mathbf{x})$$

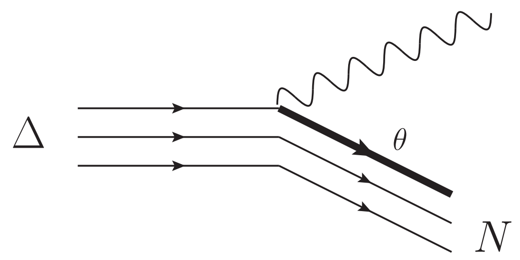
Measuring three-point functions:

$$R(t',t) = \langle 0|\Delta(t')J(0)\bar{N}(0)|0\rangle\ , \qquad S_\Delta(t)\ ,\ S_N(t)\ :\quad \text{propagators}$$

$$F = \lim_{t'\to\infty, t\to-\infty} \mathcal{N}\,\frac{R(t',t)}{S_\Delta(t'-t)}\left(\frac{S_N(t')S_\Delta(-t)S_\Delta(t'-t)}{S_\Delta(t')S_N(-t)S_N(t'-t)}\right)^{1/2}$$

Scanning the energy of Δ while keeping $\mathbf{Q}$ fixed:

- Choose $\mathbf{Q}$ along the third axis, use asymmetric boxes $L \times L \times L'$
- ... or, use (partial) twisting in the nucleon



EFT: from a finite to the infinite volume

Strong Lagrangian:

$$\mathcal{L}_{NR} = N^\dagger 2w_N(i\partial_0 - w_N)N + \pi^\dagger 2w_\pi(i\partial_0 - w_\pi)\pi$$
$$+ \; C_0 N^\dagger N \pi^\dagger \pi + X_i(\mathcal{O}_i^\dagger N\pi + \text{h.c.}) + \text{terms with derivatives}$$

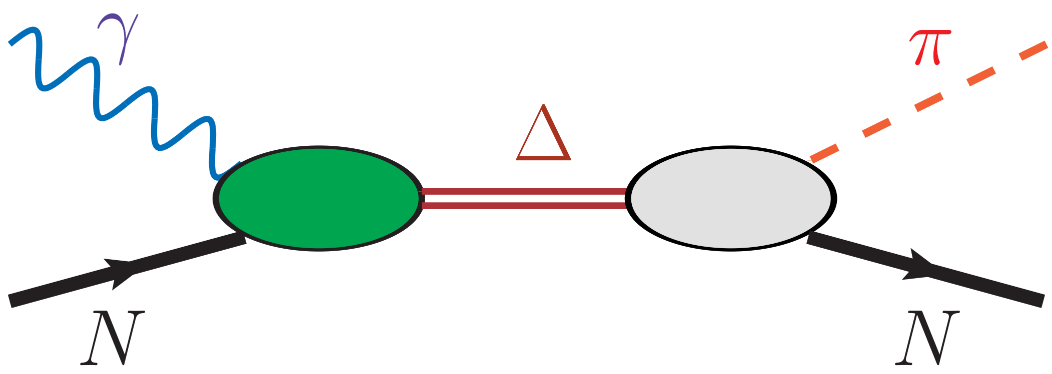
$$w_N = \sqrt{m_N^2 - \triangle}\,, \quad w_\pi = \sqrt{M_\pi^2 - \triangle}$$

- LECs $C_0, \ldots$ are of unnatural size due to the presence of the Δ
- Electromagnetic interactions: $\partial_\mu \to \partial_\mu - ieA_\mu$

↪ Calculate matrix element in EFT in a finite and in the infinite volume

↪ Establish the relation between these two quantities

Photoproduction amplitude



In the narrow width approximation. . .

$$|\text{Im}\,\mathcal{A}(\gamma^* N \to \pi N)| = \sqrt{\frac{8\pi}{p_A \Gamma}}\,|\langle\Delta|J(0)|N\rangle|\,, \qquad \delta(p_A) = 90^\circ$$

- The form factor defined on the real axis is process dependent due to the background contribution
- Analytic continuation to the resonance pole is needed!

I.G. Aznaurian *et al.*, arXiv:0810.0997; D. Drechsel *et al.*, NPA 645 (1999) 145; R.L. Workman *et al.*, PRC 87 (2013) 068201

Extraction of the photoproduction amplitude

Watson's theorem, infinite volume:

$\hookrightarrow$ Lüscher-Lellouch formula for the photoproduction amplitude:

$$\mathcal{A}_i(p_n,|\mathbf{Q}|)=e^{i\delta(p_n)}V^{1/2}\left(\frac{1}{|\delta'(p_n)+L/2\pi\,\phi'(q_n)|}\,\frac{p_n^2}{2\pi}\right)^{-1/2}|F_i(p_n,|\mathbf{Q}|)|$$

Form factor, extracted from the photoproduction amplitude:

$$|\mathrm{Im}\,\mathcal{A}_i(p_A,|\mathbf{Q}|)|=\sqrt{\frac{8\pi}{p_A\Gamma}}\,|F_i^A(p_A,|\mathbf{Q}|)|\,,\qquad \delta(p_A)=90^\circ$$

Form factor at the resonance pole is extracted through analytic continuation by using effective range expansion

Conclusions, outlook

- Effective field theories represent a powerful tool for the extraction of the physical observables (defined in the infinite volume) from the lattice data (taken in a finite volume)
- Using Chiral Perturbation theory in a finite volume, it is possible to extract the position of the resonances even in the coupled-channel cases. Using (partially) twisted b.c. allows one to enhance the sensitivity of the fit to lattice data
- Studying the dependence of the energy of the $D_{s0}^*(2317)$ meson on the twisting angle $\Rightarrow$ the w.f. renormalization constant Z $\Rightarrow$ the size of the molecular component in $D_{s0}^*(2317)$
- We define the procedure of extraction of the $\Delta N\gamma^*$ form factor on the lattice in case of unstable Δ. Analog of the Lüscher-Lellouch formula for the photoproduction amplitude is derived
- Further applications and plans: spectrum of the XYZ states on the lattice, decays of B mesons with K^* in the final state, three-particles in a finite volume

spare: Projecting out the form factors

$$
\begin{aligned}
G_2: \quad \Delta_{3/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\Delta^1 - i\Sigma_3\Delta^2) \\
G_1: \quad \Delta_{1/2} &= \frac{1}{2}(1-\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\Delta^1 + i\Sigma_3\Delta^2) \\
G_1: \quad \tilde{\Delta}_{1/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\Delta^3 \\
N_{\pm 1/2} &= \frac{1}{2}(1\pm\Sigma_3)\frac{1}{2}(1+\gamma_4)N \\
J^{\pm} &= \frac{1}{2}(J^1 \pm iJ^2)
\end{aligned}
$$

$$
\begin{aligned}
\langle\tilde{\Delta}(1/2)|J^3(0)|N(1/2)\rangle &\to A\frac{E_R - Q^0}{E_R}\,G_C(t) \\
\langle\Delta(1/2)|J^+(0)|N(-1/2)\rangle &\to A\frac{1}{\sqrt{2}}\,(G_M(t) - 3G_E(t)) \\
\langle\Delta(3/2)|J^+(0)|N(1/2)\rangle &\to A\sqrt{\frac{3}{2}}\,(G_M(t) + G_E(t))
\end{aligned}
$$