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PHYSICS OF SINGULAR POTENTIALS

I. Statement of problem (Scrhrodinger equation.Discrete Spectrum)

From the demand, that H and $P_r = -i\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$ operators are Hermitian it follows, that [1.D.Blockincev, 2.V.Pauli, 3.A.Messia]

$$\lim_{r\to 0} rR = u(0) = 0 \qquad (1.1)$$

For **regular** potentials

$$\lim_{r\to 0} r^2 V = 0 \qquad (1.2)$$

$$\underset{r\to 0}{R} = ar^l + br^{-(l+1)} \qquad (1.3)$$

Second term in (1.3) doesn't obey (1.1) and is **neglected** usually

Singular potentials

$$\lim_{r\to 0} r^2 V \to \pm\infty \qquad (1.4)$$

Transition potentials

$$\lim_{r\to 0} r^2 V \to \pm V_0 (V_0 > 0) \qquad (1.5)$$

Theorem. For attractive transition potentials Schrodinger equation except standard solutions, may also have additional solutions. *Proof:*

$$u'' + 2m\left[E - V(r)\right]u - \frac{l(l+1)}{r^2}u = 0; u = Rr \qquad (1.6)$$

At small r from (1.6) we obtain

$$\underset{r\to 0}{u} = a_{st} r^{1/2+P} + a_{add} r^{1/2-P} = u_{st} + u_{add} \qquad (1.7)$$

Where

$$P = \sqrt{\left(l + 1/2\right)^2 - 2mV_0} \qquad (1.8)$$

For

$$0<P<1/2 \qquad (1.9)$$

Both standard and additional solutions **satisfy** (1.1) condition (For **P>1/2** only **standard** solutions stay!)

From (1.8) and (1.9) we obtain condition of **existence** of additional states

$$l(l+1)<2mV_0 \qquad (1.10)$$

u_{add} satisfy requirement, that [4.-L.Schiff. Quantum mechanics.] integral from particle coordinate probability density is finite!

Remark:

In paragraph 35- "Falling on the center" [5-L.Landau, E.Lifchitz. Quantum mechanics]. behavior of $R=\frac{u}{r}$ is considered at small r

$$\underset{r\to 0}{R} = Ar^{-1/2+P} + Br^{1/2-P} \qquad (1.11)$$

In (1.11) for P<1/2 both terms are singular (second term is more singular!) and in [6- R.Newton monograph] author notice: "If P<1/2, then the second solution is irregular in sense, that it is dominant above first solution". So R.Newton come very close to additional state problem, but doesn't mention that they exist! In [5] potential is made regular by cutting off it at some small r_0 and the limit $r_0 \to 0$ is taken, which selects less singular solution and so additional solutions are neglected! But if we multiple (1.11) relation on r we get (1.7) relation, where we have, no singularity in the 0<P<1/2 region and as mentioned above standard and *additional solutions* are "equal in rights" members of (1.7) relation!

I I. Introduction of self-adjoint extension parameter

In [7-K.Case.Phys.Rev.80,797(1950); 8-K.Meetz.Nuovo Cimento 34, 690(1964); 19-A.Perelomov, V.Popov.TMF.vol 4 (1970)] was shown, that for attractive regular and transition potentials it isn't enough to know potential and is necessary to introduce one arbitrary constant, which is equivalent to give boundary condition at the origin. Indeed, when

$$2mV_0 > (l+1/2)^2 \tag{2.1}$$

P is complex from (1.8), both u_{st} and u_{add} solutions have same behavior at the origin and for example for $\lim_{r\to 0} V = -\frac{V_0}{r^2}$ at small distances one have [5, 7]

$$u \approx A\sqrt{r}\cos\left(\sqrt{2mV_0 - (l+1/2)^2}\ln r + B\right) \tag{2.2}$$

B is arbitrary constant. On the Mathematical language it means, that H is Hermitian (symmetric), but isn't Self-adjoint operator and it is necessary to introduce 1 parameter to make H Self-adjoint ![M.Reed,B.Simon:vol 2]. As was shown in [7] if B is fixed constant, then all eigensolutions form a complete orthonormal set, and E-eigenvalues are real! But in this case we have"falling" on the center and energy isn't bounded from below!

In the region

$$2mV_0 < (l+1/2)^2 \tag{2.3}$$

based on the above mentioned paragraph of [5] , u_{add} solutions is neglected. We notice above, that in the 0<P<1/2 region it is necessary to preserve u_{add} !

Then for arbitrary E_1 and E_2 levels ortogonality condition is

$$m(E_2 - E_1)\int_0^\infty u_2 u_1 dr = 2P\left\{a_1^{st} a_2^{add} - a_2^{st} a_1^{add}\right\} \tag{2.4}$$

And for ortogonality right side of (2.4) is zero

$$\frac{a_1^{st}}{a_1^{add}} = \frac{a_2^{st}}{a_2^{add}} \tag{2.5}$$

So, we get it is necessary to introduce self-adjoint τ extension parameter

$$\tau = \frac{a_{add}}{a_{st}} \tag{2.6}$$

From (1.7) and (2.6) we have:

a) $a_{add} = 0(\tau = 0)$. We retain only standard levels

b) $a_{st} = 0(\tau = \pm\infty)$. We retain only additional levels

c) When $\tau \neq \pm\infty, 0$, then both levels exist at the same time!

For some unknown reasons the Nature choose only standard levels yet! We think,
that other cases are also possible!

I I.I Model of Valent electron

$$V = -\frac{V_0}{r^2} - \frac{\alpha}{r}; V_0, \alpha > 0 \tag{3.1}$$

This potential "naturally" appears for coulomb potential in the Klein-Gordon equation. Following [12-W.Krolikowski; Bulletin De L' academics polonaise Vol XVII.83(1979);13-A.A.Khelashvili,T.P.Nadareishvili, Bulletin of Georgian Acad.Sci:Vol 164.no1(2001)] we obtain general solution for (3.1) potential

$$u = C_1 \rho^{1/2+P} e^{-\rho/2} F(1/2 + P - \lambda, 1 + 2P; \rho) + C_2 e^{1/2-P} e^{-\rho/2} F(1/2 - P - \lambda, 1 - 2P; \rho) \tag{3.2}$$

Where P is given again by (1.8) and

$$\rho = \sqrt{-8mE} \cdot r; \lambda = \frac{2m\alpha}{\sqrt{-8mE}}; E < 0 \tag{3.3}$$

we get transcendental equation for E

$$\frac{\Gamma(1/2 - \lambda - P)}{\Gamma(1/2 - \lambda + P)} = -\frac{\tau}{(-8mE)^P} \frac{\Gamma(1 - 2P)}{\Gamma(1 + 2P)} \tag{3.4}$$

where SAE parameter τ is

$$\tau = \frac{C_2}{C_1} \frac{1}{(-mE)^P} \tag{3.5}$$

E depends on τ parameter. For $\tau = 0$ and $\tau = \pm\infty$ we obtain standard and additional levels analitically

$$E_{st,add} = -\frac{m\alpha^2}{2[1/2 + n_r \pm P]^2} = -\frac{m\alpha^2}{2\left[1/2 + n_r \pm \sqrt{(l+1/2)^2 - 2mV_0}\right]^2} \tag{3.6}$$

$$n_r = 0,1,2$$

Remark: For $V_0 < 0$ in (3.1), we get Kratzer Molecular potential and we obtain for standard levels well known formula, but in this case isn't fulfilled (1.10) condition and so we have no additional levels for Kratzer potential .

For alkaline metal atoms (Li,Na,K,Rb,Cs) is used (3.1) potential [14-S.Frish .Optical specra of atoms;15 –M.Eliashevich.Atomic and molecular spectroscopy].Spectra of this atoms is similar hydrogen atom spectra

$$E_{n'} = -R\frac{1}{n'^2} \tag{3.7}$$

Where R is Rydberg constant and n' is effective principal number

$$n' = n_r + l' + 1 \tag{3.8}$$

And l' is defined from

$$l'(l'+1) = l(l+1) - 8mV_0 \tag{3.9}$$

For l' is taken only + sign in front of root P [12,13]

$$l' = -1/2 + P = -1/2 + \sqrt{(l+1/2)^2 - 2mV_0} \tag{3.10}$$

So up to now wasn't considered additional levels (- sign in front of root). Then in [13] the root is expand is expand for small V_0

$$E_{st} = -R\frac{1}{\left(n+\Delta_l^{st}\right)^2};\, n = n_r + l + 1 \tag{3.11}$$

Where Δ_l^{st} is Rydberg correction (quantum defect)

$$\Delta_l^{st} = -\frac{2mV_0}{2l+1} \tag{3.12}$$

For E_{add} we can't take small V_0, because $l(l+1) < 2mV_0$ So for E_{st} at $V_0 \to 0$ one get hydrogen atom spectra; E_{add} exist only for "strong" values of V_0!

So it is expectable, that in the Model of Valent electron, beside the well known E_{st} levels, may also exist E_{add} and (3.4) transcendental equation levels.

Remark: Our formalism works everywhere, where (3.1) potentials works: for excited (Rydberg) atoms, for alkaline isoelectronic ions and etc.

I V . Singular (Spiked) Oscillator model

$$V = -\frac{V_0}{r^2} + gr^2;\; V_0, g > 0 \tag{4.1}$$

Use: Calogero model, Fractional statistics and anyons, Quantum Hall effect, Spin chains, Two dimensional QCD.

$$\frac{\Gamma(-1/4\sqrt{2m/g}\,E + 1/2 - P/2)}{\Gamma(-1/4\sqrt{2m/g}\,E + 1/2 + P/2)} = -\frac{\tau}{\left(2mg\right)^{P/2}}\frac{\Gamma(1-P)}{\Gamma(1+P)} \tag{4.2}$$

□

For $\tau = 0$ and $\tau = \pm\infty$ we get standard and additional levels

$$E_{st,add} = 2\sqrt{g/2m}\left\{2n_r + 1 \pm P\right\};\; n_r = 0,1,2... \tag{4.3}$$

Remark: For $V = -V_0/r^2 + W(r)$ potential (where W is regular potential) we can define generally quantum defect by

$\Delta_l^{st} = P - (l + 1/2)$ as a deviation from W(r), because when $V_0 = 0$, then P=l+1/2 and $\Delta_l^{st} = 0$.

V. Scattering Problems (Continuous Spectrum).

$$V = -\frac{V_0}{r^2}; V_0 > 0 \qquad (5.1)$$

This interaction is realized in nature- physical applications: .

1). Gharge interacting with a point dipole [14H.Camblong...Phys.Rev. Lett. 87, 220402 (2001)]

2). Interaction of a neutral, but polarizable atom with a charged wire [15-J.Denschlag; Phys.Rev.Lett.81.737. (1998)

3) Aaronov-Bom effect [16 –J.Audretsh…J.Phys.A28,2359 (1995)].

4). Black holes [Gupta,Shabad…]

$$U_k(r) = \sqrt{kr}\left\{A(k)J_P(kr) + B(k)J_{-P}(kr)\right\}; k^2 = 2mE; E > 0 \qquad (5.2)$$

In (5.2) for 0<P<1/2 $\sqrt{r}J_{-P}(kr)$ is regular at the origin and we keep it!

a). Introduction of SAE parameter

$$I = \int_0^\infty r^2 R_{k'}^*(r) R_k(r) dr = 2\pi\delta(k' - k); R_k = \frac{u_k}{r} \qquad (5.3)$$

We use integrals from [17-J.Audretsch.J.Phys.A34,235 (2001)] for Bessels functions and get

$$I = \left\{AA^* + BB^* + (B^*A + A^*B)\cos\pi P\right\}\delta(k' - k) + \frac{2\sin\pi P}{\pi(k^2 - k'^2)}\left\{\left(\frac{k}{k'}\right)^P B^*(k')A(k) - \left(\frac{k}{k'}\right)^{-P} A^*(k')B(k)\right\}$$

(5.4)

$$\frac{B^*(k')}{A^*(k')}(k')^{-2P} = \frac{B(k)}{A(k)}k^{-2P} = \tau_P$$

(5.5)

(5.5) is analog of (2.5) for continuous spectrum.

From (5.4) and (5.5) we get

$$AA^*\left[\tau_P^2 k^{4P}+2\tau_P k^{2P}\cos\pi P+1\right]=2\pi$$

(5.6)

Based on the methodology of [17] and [18 - S.Alliluev.JETP.Vol 61,p15 (1970)] articles,where is considered

$$I=\lim_{R\to\infty}\int_0^R u_{k'}^*(r)u_k(r)=\frac{1}{k'^2-k^2}\left[u_{k'}^*\frac{du_k}{dr}-u_k^*\frac{du_{k'}}{dr}\right]_0^R \qquad (5.7)$$

One can show, that τ parameter is introduced from the lower limit of the (5.7) integral as it was for the bound states and (5.6) is introduced from the upper limit of this integral (For bound states wave function decrease at large distances and we no analog of (5.6) relation).

b).Phase Shifts Calculation.

$$u_k(r)=\sqrt{kr}\left\{A(k)+B(k)N_P(kr)\right\}$$

(5.8) in (5.8) Second term is regular at the origin for 0<P<1/2 and we keep it!

$$\lim_{r\to\infty}R=\frac{C}{r}\sin\left(kr-l\pi/2+\delta_l\right);R=\frac{u}{r}$$

(5.9)

$$\delta_P=\left[l+1/2-P\right]\pi/2-arctgB/A$$

(5.10)

or using (5.5) definition of SA parameter we get:

$$\delta_P=\left[l+1/2-P\right]\pi/2-arctg(\tau_P k^{2P}) \qquad (5.11)$$

In the literature is known only first term [17.A.M.Perelomov; V.S.Popov.TMF. Vol 4.No1(1970)]

a).B=0; $\tau_P=0;\delta_P^{st}=\left[l+1/2-P\right]\pi/2$ (5.12)

b).A=0; $\tau_P=\pm\infty;\delta_P^{add}=\delta_l^{st}\pm\pi/2$ (5.13)

+ sign in (5.13) is excluded, comparing it with asymptotic expression

$$N_P(kr) \approx \sqrt{\frac{2}{\pi kr}} \sin\left(kr - P\pi/2 - \pi/4\right)$$

Remarks: 1. From (5.11) we see that δ_P is depended on the energy $\left(k^2 = 2mE\right)$for $\tau_P \neq 0, \infty$,so scale invariance is violated!

2. We considered $V = -\frac{V_0}{r^2}$attractive potential, for which $\delta_P > 0$.As one see from (5.11) may be $\delta_P < 0$or we get repulsive potential! So τ_P parameter may change the NATURE of potential! We have two possibilities: a).From the Physical motivation restrict $\tau_P$parameter (Don't change attractive potential by repulsive!) or as one see from (5.11) demand

$$\left[l + 1/2 - P\right]\pi/2 - arctg\left(\tau_P k^{2P}\right) > 0$$

b).Agree, that $\tau_P > 0$can change potential nature!

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 ; f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(S_l - 1\right)P_l\left(\cos\theta\right) \qquad (5.14)$$

$$S_l = e^{2i\delta_l} \qquad (5.15)$$

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \qquad (5.16)$$

$$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty} f_l P_l(\cos\theta) \qquad (5.17)$$

$$f_l = \frac{1}{2ik}\left(S_l - 1\right) = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) \qquad (5.18)$$

$$\sigma_l = 4\pi\left(2l+1\right)\left|f_l\right|^2 \qquad (5.19)$$

Remarks: 1.$l(l+1) < 2mV_0$ (1.10).So in (5.14 – 5.19) one need SAE for l ,which satisfy (1.10).(l=0 always satisfy it!).

2. Total cross section σ is infinite for $V = -\frac{V_0}{r^2}$in usual quantum

mechanics $(\tau=0)$.We show, that for A=0 (τ=+∞) and small k σ is again infinite, but in general case, when
$\delta_P=[l+1/2-P]\pi/2-arctg\left(\tau_P k^{2P}\right)$ − σ can become finite! This problem needs more careful investigation!

VI.Scattering length a

In [17] is calculated scattering length a for the following potential in the l=0 state

$$V(r)=-\frac{V_0}{r^2}\theta(R-r) \tag{6.1}$$

We now obtain more general formula using SAE.When r<R wave function is

$$\chi=\begin{cases} Ar^{1/2+P}+Br^{1/2-P};2mV_0<1/4;P=\sqrt{1/4-2mV_0} \\ r^{1/2}\sin(\nu\ln r+\gamma_0);2mV_0>1/4;\nu=\sqrt{2mV_0-1/4} \end{cases} \tag{6.2}$$

γ_0 is SAE parameter, when one have “falling” on the center and is known in the literature [5,17]

For r>R

$$\chi_0=C(r-a) \tag{6.3}$$

“Sewing” condition at r=R gives

$$a=-R\frac{(1-2P)AR^P+BR^{-P}(1+2P)}{(1+2P)AR^P+BR^{-P}(1-2P)};2mV_0<1/4 \tag{6.4}$$

$$a=-R\frac{1-2\nu ctg(\nu\ln R-\gamma_0)}{1+2\nu ctg(\nu\ln R-\gamma_0)};2mV_0>1/4 \tag{6.5}$$

When B=0 we get [17] article formula

$$a=-R\frac{1-2P}{1+2P} \tag{6.6}$$

As P<1/2, a<0 and it corresponds to attractive potential, but from (6.4) a have no definite sign- we can’t say one have attractive or repulsive interaction!

SAE now we define

$$\tau = -A/B \tag{6.7}$$

$$a = \frac{a_1\tau + b_1}{a_2\tau + b_2} \tag{6.8}$$

$$\begin{gathered} a_1 = -(1-2P)R^{1+P}; b_1 = (1+2P)R^{1-P} \\ a_2 = (1+2P)R^{P}; b_2 = -(1-2P)R^{-P} \end{gathered} \tag{6.9}$$

In the region

$$\tau_1 < \tau < \tau_2 \tag{6.10}$$

where

$$\tau_1 = \frac{1-2P}{1+2P}\frac{1}{R^P}; \tau_2 = \frac{1+2P}{1-2P}\frac{1}{R^{2P}}$$

a>0 and we have repulsive interaction! $V(r) = -\frac{V_0}{r^2}\theta(R-r)$ is attractive potential and τ parameter from (6.10) region can change it NATURE! Again one have two alternatives : a). From the physical motivation exclude (6.10) region. b). Agree that τ can change interaction nature!

Remarks: 1). We expand (6.4) and (6.5) near $2mV_0 = 1/4$ and get relation between τ and γ_0

$$2ctg\gamma_0 = \frac{\tau+1}{\tau-1} \tag{6.11}$$

2). $\sigma_{tot} = 4\pi a^2(\tau) = \sigma(\tau)$ depends on ! $\sigma > 0$ demand restrict τ!

VII.Scattering effective radios ($2mV_0 < 1/4$)

$$r_0 = 2\int_0^\infty \left[u_0^2(r) - \chi_0^2(r)\right]dr \tag{7.1}$$

where

$$u_0 = C(r-a) \qquad (7.2)$$

$$\chi_0 = \begin{cases} Ar^{1/2+P} + Br^{1/2-P}; r < R \\ C(r-a); r > R \end{cases} \qquad (7.3)$$

$$r_0 = 2/3C^2\left\{(R-a)^3 + a^3\right\} - D^2\left\{\frac{\tau^2}{2(p+1)}R^{2P+2} - \frac{R^{2(1-P)}}{2(1-P)} + \tau R^2\right\}$$

$$D = \frac{C(R-a)}{R^{1/2-P} - \tau R^{1/2+P}} \qquad (7.4)$$

$r_0 > 0$ demand restrict τ!

VIII.Modification of Rutherford formula

For Model of valence electron (3.1) potential for scattering case we get

$$u = C_1\rho^{1/2+P}e^{-\rho/2}F(1/2+P-\lambda, 1+2P;\rho) + C_2\rho^{1/2-P}e^{-\rho/2}F(1/2-P-\lambda, 1-2P;\rho)$$

(8.1)

Where P is given again by (1.8) and

$$\rho = 2ikr; \lambda = -i\frac{m\alpha}{k} = -i\eta; E > 0; k = \sqrt{2mE}; \eta = \frac{m\alpha}{k}; \qquad (8.2)$$

SAE parameter is

$$\tau = \frac{C_2}{C_1}(2ik)^{-2P} \qquad (8.3)$$

$$\lim_{r\to\infty} u = \sin\left[kr + \eta\ln 2kr - (P-1/2)\pi/2 - \delta_{coul}^{st} + \delta_P\right] \qquad (8.4)$$

Where

$$\delta_{coul}^{st} = \arg\Gamma(1/2+P-\lambda) \qquad (8.5)$$

$$\delta_P = arctg\Psi; \Psi = \tau_P(2k)^{2P}\frac{\Gamma(1-2P)\left|\Gamma(1/2+P-\lambda)\right|}{\Gamma(1+2P)\left|\Gamma(1/2-P-\lambda)\right|} \qquad (8.6)$$

$$\delta = \delta_P - \delta_{coul}^{st} + [l+1/2-P] \qquad (8.7)$$

$$S_P = e^{i\pi[l+1/2-P]} \frac{\Gamma(1/2+P+\lambda)}{\Gamma(1/2-P-\lambda)} e^{2iarctg\Psi} \qquad (8.9)$$

$$S = S_P^{st} \frac{1+i\Psi}{1-i\Psi} \qquad (8.10)$$

$i\psi = 1$ is pole!

And we again obtain (3.4) transendental equation for bound states.

$$l(l+1) < 2mV_0 \qquad (1.10)$$

$$f(\theta) = \frac{1}{2ik}\left\{\sum_{l=0}^{\left[-1/2+\sqrt{1/4+2mV_0}\right]} (2l+1) P_l(\cos\theta)\left[S_P^{st} \frac{1+i\Psi}{1-i\Psi}\right]\right\} + \sum_{\left[-1/2+\sqrt{1/4+2mV_0}\right]}^{\infty} (2l+1)P_l(\cos\theta)\left[S_P^{st} - 1\right]$$

(8.11)

$$f(\theta) = f_{SAE} + f_{VE} \qquad (8.12)$$

When $V_0 \to \infty$,in the (8.9) leading term is f_{SAE} and for small V_0 is f_{VE}

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = |f_{VE}|^2 + |f_{SAE}|^2 + 2\operatorname{Re} f_{VE}^* f_{SAE} \qquad (8.13)$$

$$P = \sqrt{(l+1/2)^2 - 2mV_0} \approx (2l+1) - \frac{2mV_0}{2l+1} \qquad (8.14)$$

We keep in this case in f_{SAE} only l=0 term and get

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\eta^2 k^2 \sin^4\theta/2}\left[1 + \frac{2(\pi V_0)^2 m^3}{E}\sin^2\theta - 2\pi m V_0\sqrt{2m/E}\cos 2(\sigma_{1/2} - \sigma_0\right] +$$
$$\frac{\sin^2\delta_{SAE}^0}{k^2} - \eta\frac{2\sin\delta_{SAE}^0}{k^3\sin^2\theta}[\cos\left(\eta\ln\sin^2\theta/2 - 2\sigma_0 - 2\pi m V_0 - \delta_{SAE}^0\right) -$$
$$\pi m V_0\sqrt{2m/E}\sin\theta\cos\left(\eta\ln\sin^2\theta/2 - 2\pi m V_0 - \delta_{SAE}^0 - 2\sigma_{-1/2}\right) \qquad (8.13)$$

where the first term is usual Rutherford formula modified for VE model last two terms are caused by SAE procedure and are similar to the short range interactions. So SAE can again play a role in potential nature! This formalism can be used also for, π^- π+scattering, where is used Klein-Gordon equation.

VIII. Concluding remarks. Summary

1. Our main result: We show, that for $\lim\limits_{r\to 0} r^2 V = -V_0 (V_0 > 0)$ potentials in the region $\left(l+1/2\right)^2 > 2mV_0$ (no "falling onto center!) it is necessary to keep second additional solution in the 0<P<1/2 interval (We have our variant of Landau mentioned paragraph!) and it is also necessary to introduce self-adjoint extension τ parameter. in both bound states and scattering problems.

2. Physical quantities E, a, r_0, σ depend on τ parameter and by this reason physical picture is different then in usual quantum mechanics! (As was mentioned above SAE can change nature of potential, δ_p became energy dependent for $V = -\dfrac{V_0}{r^2}$ potential and so on.

We have three possibilities:

1).It should be found another strong requirement in the quantum mechanic mathematical formalism, which "destroys" additional states!

2) If it isn't possible, try to ''struggle' against τ parameter by physical demands: $r_0 > 0, \sigma > 0$, don't change physical nature of interaction and so on.

3).Admit SAE existence and find new levels, and so on. And now it stay open the following question: Why the NATURE "select" only standard states $(\tau = 0)$?!